A Checked Candidate Extension of Stadlmann's Method to $H_1 \le 216$

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Abstract

We record a carefully rechecked candidate extension of Stadlmann's 2026 bounded-gaps framework to the target $H_1 \le 216$ using $k = 46$. The purpose of this note is not to claim a new theorem, but to isolate a parameter set for which all scalar inequalities and all combinatorial partition conditions in the stated equidistribution criterion can be verified explicitly, apart from known defects in the proof of that criterion itself. The remaining sieve-theoretic task is then a single variational certificate $46J(F) > I(F)$ on the stated restricted support. We also verify directly that the displayed 46-tuple of diameter 216 is admissible.

1 Status of the argument

The conclusion established below is conditional in two distinct senses.

(i) Stadlmann's manuscript contains several gaps and drafting defects in the proof of its general equidistribution criterion. In particular, the Type IIc condition is literally impossible on the stated interval $\omega_0 < 0$, and other steps in the proof of the criterion require repair. This note verifies the candidate parameters against the stated scalar conditions and gives a rigorous partition argument for the meaningful range $\omega_0 \ge 0$; it does *not* supply a complete replacement proof of the equidistribution criterion.

(ii) Even after such an analytic repair, one still needs an explicit symmetric function F on the restricted support satisfying

$$46J(F) > I(F).$$

Equivalently, after choosing a finite basis, one needs an exact rational or interval-certified quadratic-form inequality. No such certificate is claimed here.

Thus the note should be read as a checked reduction of the target $H_1 \le 216$ to two sharply identified remaining tasks, rather than as a proof of $H_1 \le 216$.

2 The admissible 46-tuple

Let

$$H_1 = \liminf_{n\to\infty}(p_{n+1} - p_n).$$

The narrow-tuples database contains the following 46-tuple of diameter 216:

$$\begin{aligned}\mathcal{H}_{46} = \{&0, 4, 6, 16, 18, 28, 30, 34, 40, 48, 54, 58, 60, 64, 70, 76, 78, 84, 88, 96,\\ &100, 106, 114, 118, 120, 126, 130, 138, 144, 148, 154, 160, 166, 168, 174, 180,\\ &184, 186, 190, 196, 198, 204, 208, 210, 214, 216\}.\end{aligned}$$

It is useful to check admissibility directly. For every prime $p \le 46$, the following table gives one residue class modulo p omitted by $\mathcal{H}_{46}$:

p	2	3	5	7	11	13	17	19	23	29	31	37	41	43
omitted residue	1	2	2	3	2	7	5	17	21	8	1	8	1	3

For primes $p > 46$, a set of only 46 residues cannot cover all p residue classes. Hence $\mathcal{H}_{46}$ is admissible. Consequently, any valid $k = 46$ Maynard–Tao/GPY criterion for two primes implies

$$H_1 \le 216.$$

3 Candidate support parameters

We use a single A-interval. To avoid confusing Stadlmann's support-enlargement parameter with the tiny technical error parameter, write

$$\varepsilon_s = 0.0075 = \frac{3}{400}, \qquad \epsilon_0 = 10^{-10}.$$

Choose

$$A = 0.2583 = \frac{2583}{10000}, \qquad \delta = 0.012 = \frac{3}{250}, \tag{1}$$

and

$$\xi_1 = 0.399, \qquad \xi_2 = 0.400, \qquad \xi_3 = 0.40001. \tag{2}$$

The strict inequality $\xi_2 < \xi_3$ is intentional because the proof of the Harman decomposition in Stadlmann's manuscript invokes a strict inequality at one point.

For the large-coordinate restrictions set, for all indices for which B_m is needed,

$$B_1 = B_2 = 0.15, \qquad B_m = 0.16 \quad (m \ge 3). \tag{3}$$

The structural condition on B_m is satisfied:

$$\delta < B_m, \qquad B_m \le B_{m+1} \le B_m + \delta,$$

whenever both sides are defined, since the only nonzero jump is

$$0.16 - 0.15 = 0.01 < 0.012 = \delta.$$

For bookkeeping in zero-factor cases one may set $B_0 = 0$.

With one A-interval, the candidate support is

$$\mathcal{T}_{46} := \left\{ t \in [0,1]^{46} : 0 \le \sum_{i=1}^{46} t_i < A + \varepsilon_s = 0.2658, \quad \sum_{i:t_i>\delta} t_i \le B_{|\{i:t_i>\delta\}|} \right\}. \tag{4}$$

4 Scalar equidistribution inequalities

We now check the scalar hypotheses appearing in Stadlmann's stated equidistribution criterion. All numerical values below include the tiny ϵ_0 terms exactly as written in the proposition.

4.1 Type I

The first Type I quantity is

$$\begin{aligned}\xi_1 - 4A + \frac{2}{3} - 2\epsilon_0 &= 0.399 - 4(0.2583) + \frac{2}{3} - 2 \cdot 10^{-10} \\ &= 0.032466666466\ldots > 0.012 = \delta.\end{aligned}$$

The second is

$$\frac{9}{7} - \frac{34}{7}A - 2\epsilon_0 = 0.031114285514\ldots > 0.012.$$

Thus the Type I scalar condition holds.

4.2 Type II

First,

$$\begin{aligned}\frac{19}{2} - 36A - 13\delta + 100\epsilon_0 &= 9.5 - 36(0.2583) - 13(0.012) + 10^{-8} \\ &= 0.04520001 > 0.\end{aligned}$$

The two quantities in the second Type II condition are

$$\begin{aligned}\frac{\xi_2}{10} - \frac{32A}{10} + \frac{8}{10} - 2\epsilon_0 &= 0.0134399998 > 0.012, \\ \frac{\xi_2}{4} + \frac{11}{16} - 3A - 2\epsilon_0 &= 0.0125999998 > 0.012.\end{aligned}$$

The latter is the tightest scalar inequality; its slack is

$$0.0005999998.$$

4.3 Type III

Finally,

$$\frac{11}{8} - \frac{7}{2}A - \frac{9}{8}\xi_3 - 2\epsilon_0 = 0.0209387498 > 0.012.$$

Thus all three stated scalar conditions hold.

5 Harman-sieve parameters

The inequalities appearing in Stadlmann's Harman-sieve proposition are

$$2\xi_1 + 3\xi_2 < 2, \qquad \xi_2 \le \xi_3, \qquad \xi_1 + 9\xi_2 < 4,$$

$$2\xi_1 + \xi_2 > 1, \qquad 17\xi_2 < 7.$$

For (2),

$$\begin{aligned} 2\xi_1 + 3\xi_2 &= 1.998 < 2, \\ \xi_2 &= 0.4 < 0.40001 = \xi_3, \\ \xi_1 + 9\xi_2 &= 3.999 < 4, \\ 2\xi_1 + \xi_2 &= 1.198 > 1, \\ 17\xi_2 &= 6.8 < 7. \end{aligned}$$

Moreover $\xi_2 = 0.4$. In Stadlmann's construction, at this endpoint the two negative Buchstab pieces are empty, so the intended minorant is simply

$$\rho(n; x) = 1_{\mathbb{P}}(n), \qquad c_1 = c_2 = 0. \tag{5}$$

The roughness exponent supplied by the same construction is

$$\beta = 1 - 2\xi_2 = 0.2 > B_1 = 0.15,$$

so the roughness hypothesis in the GPY proposition is compatible with (3).

6 Partition conditions

Put

$$\omega = A - \frac{1}{4} = 0.0083.$$

Let

$$(y_1, \ldots, y_{m+m'}) \in \Xi(B_m, B_{m'}, m, m', \delta).$$

For $m, m' \geq 1$ one always has

$$S := \sum_i y_i \leq B_m + B_{m'} \leq 0.32. \tag{6}$$

If one of m, m' is zero, the stronger bound $S \leq 0.16$ holds, so all arguments below become easier.

6.1 Condition A: Type I

Take I_1 to be the full index set and $I_2 = \varnothing$. The two capacities are

$$\begin{aligned} \xi_1 - 2\epsilon_0 &= 0.3989999998 > 0.32, \\ \frac{1}{6} - 4\omega - 2\epsilon_0 &= 0.133466666466\ldots > 0. \end{aligned}$$

Thus condition A holds.

6.2 Condition B: Type IIa

Again put everything in I_1. The capacities are

$$\begin{aligned} \frac{2}{5} + \frac{24}{5}\omega + \frac{7}{5}\delta - 2\epsilon_0 &= 0.4566399998 > 0.32, \\ \frac{1}{14} - \frac{24}{7}\omega - 2\epsilon_0 &= 0.042971428371\ldots > 0. \end{aligned}$$

Thus condition B holds.

6.3 Condition C: Type IIb

The three capacities are

$$\begin{aligned} C_1 &= \frac{1}{3} + 8\omega + \frac{7}{3}\delta - 4\epsilon_0 = 0.427733332933\ldots, \\ C_2 &= \frac{1}{10} - \frac{34}{5}\omega - \frac{7}{5}\delta - 4\epsilon_0 = 0.0267599996, \\ C_3 &= \frac{1}{35} + \frac{22}{35}\omega + \frac{21}{35}\delta - 4\epsilon_0 = 0.040988571028\ldots. \end{aligned}$$

Hence $C_1 > 0.32$ and the unused capacities are positive. Taking the full set as I_1 proves condition C.

6.4 Condition D: Type IIc for the meaningful range $\omega_0 \geq 0$

The manuscript states this condition for

$$\omega_0 \in [-\epsilon_0, \omega]$$

and simultaneously requires

$$\sum_{i \in I_4} y_i \leq 8\omega_0.$$

For $\omega_0 < 0$ this is impossible even for $I_4 = \varnothing$. Therefore the proposition, as printed, cannot be literally correct. We do *not* claim here that ordinary Bombieri–Vinogradov by itself automatically closes the entire interval of moduli arbitrarily close to $x^{1/2}$ from below. A corrected proof of the analytic criterion must address this point.

For the meaningful Type IIc range

$$0 \leq \omega_0 \leq \omega = 0.0083, \tag{7}$$

we can nevertheless verify the required partition rigorously. Let

$$\xi_2 - \epsilon_0 \leq \gamma \leq \frac{1}{3} + 8\omega + \frac{7}{3}\delta + 3\epsilon_0.$$

Define the first two capacities

$$D_1 = \gamma - 2\delta - 8\omega_0 - \epsilon_0, \qquad D_2 = \frac{1}{2} - \gamma - 2\omega_0 - \epsilon_0.$$

The minimum possible value of D_1 is

$$\begin{aligned} D_{1,\min} &= (\xi_2 - \epsilon_0) - 2\delta - 8\omega - \epsilon_0 \\ &= 0.3095999998. \end{aligned}$$

If $S \leq D_1$, put all indices in I_1 and leave I_2, I_3, I_4 empty.

Suppose now that $S > D_1$. Since $S \leq 0.32$,

$$0 < S - D_1 \leq 0.32 - 0.3095999998 = 0.0104000002 < \delta. \tag{8}$$

If both large-coordinate groups had sizes at most 2, then by (3) we would have $S \leq 0.15 + 0.15 = 0.30 < D_1$, a contradiction. Hence at least one group has size at least 3. That group has total at most 0.16, so it contains an element y with

$$\delta \leq y \leq \frac{0.16}{3} = 0.053333\ldots. \tag{9}$$

Because of (8), moving this one element from I_1 to I_2 makes the remaining I_1-sum at most D_1.

It remains to verify that $y \le D_2$. From $S > D_1$ we have

$$\gamma < S + 2\delta + 8\omega_0 + \epsilon_0.$$

Consequently

$$\begin{aligned} D_2 &> \frac{1}{2} - S - 2\delta - 10\omega_0 - 2\epsilon_0 \\ &\ge \frac{1}{2} - 0.32 - 0.024 - 10(0.0083) - 2\epsilon_0 \\ &= 0.0729999998. \end{aligned}$$

Together with (9), this gives $y < D_2$. Taking $I_3 = I_4 = \varnothing$ is allowed in the range (7), since

$$4\omega_0 + \delta - \epsilon_0 > 0, \qquad 8\omega_0 \ge 0.$$

Thus condition D is rigorously verified for every $\omega_0 \in [0, 0.0083]$.

6.5 Condition E: Type III

The two capacities are

$$\begin{aligned} 1 - 6\omega - \frac{3}{2}\xi_3 - 2\epsilon_0 &= 0.3501849998 > 0.32, \\ \frac{5}{2}\omega + \frac{3}{8}\xi_3 - 2\epsilon_0 &= 0.1707537498 > 0. \end{aligned}$$

Hence condition E holds by putting every index in I_1.

7 What the analytic check actually proves

The calculations above establish the following statement without numerical optimization.

Proposition 1 (Checked parameter reduction)**.** *For the parameter choices* (1), (2), *and* (3), *all scalar conditions in Stadlmann's stated equidistribution criterion hold. Conditions A, B, C and E of its partition hypothesis hold for every relevant tuple. Condition D holds for every* $\omega_0 \in [0, 0.0083]$*. The zero-factor cases become trivial after the natural convention* $B_0 = 0$*. The printed negative-*ω_0 *range remains a defect that must be repaired at the analytic level.*

This is the point at which one must distinguish a parameter check from a proof of equidistribution. Earlier auditing of the manuscript found additional proof-level gaps in the general criterion (including issues in the Type I, Type IIc and Type III reductions). Proposition 1 therefore verifies that the candidate parameters are compatible with the *intended* criterion; it does not validate those unresolved analytic steps.

8 The remaining variational certificate

Assume now that the GPY proposition and the equidistribution machinery have been repaired so that they apply to the support (4). By (5), the sieve inequality becomes simply

$$46J(F) > I(F) \tag{10}$$

for some symmetric square-integrable F supported on $\mathcal{T}_{46}$.

After choosing a finite symmetric basis $G_1, \ldots, G_r$ and writing

$$F = \sum_{j=1}^{r} c_j G_j,$$

let M_1 and M_2 denote the exact Gram matrices representing I and J. A rigorous completion does not require trusting a floating-point eigenvalue: it is enough to exhibit one rational vector $c \neq 0$ satisfying

$$\boxed{c^T(46M_2 - M_1)c > 0.} \tag{11}$$

All matrix entries and the final quadratic form should be evaluated exactly or enclosed by rigorous intervals.

8.1 Unrestricted enlarged-simplex benchmark

If one temporarily removes the B_m restriction and keeps only

$$\sum t_i \leq A + \varepsilon_s,$$

then the scaling $t_i = Au_i$ produces the Polymath epsilon-enlarged problem with

$$\eta = \frac{\varepsilon_s}{A} = \frac{0.0075}{0.2583} = \frac{25}{861} = 0.029036\ldots.$$

In that unrestricted problem, (10) would follow from

$$M_{46,25/861} > \frac{1}{A} = \frac{10000}{2583} = 3.871467\ldots. \tag{12}$$

This is only a benchmark: the actual function must obey the B_m cutoff in (4). Numerical experiments can guide the choice of basis, but they are not a substitute for (11).

9 Conditional candidate conclusion

Theorem 2 (Conditional candidate for $H_1 \leq 216$)**.** *Assume the following two statements.*

(a) Stadlmann's GPY/equidistribution framework is repaired rigorously, including the negative-ω_0 defect and the other proof-level gaps identified in the manuscript, in a form applicable to the support (4) *with the parameters* (1), (2), *and* (3).

(b) There exists a symmetric square-integrable function F supported on $\mathcal{T}_{46}$ for which $46J(F) > I(F)$; equivalently, an exact finite-dimensional certificate of the form (11) *is produced.*

Then

$$\boxed{H_1 \leq 216}.$$

Proof. Under assumption (a), the prime indicator $\rho = 1_{\mathbb{P}}$ has the required equidistribution properties for the relevant moduli and satisfies the hypotheses of the corrected GPY proposition. By assumption (b), the GPY variational inequality is strict for $k = 46$. Hence infinitely many translates of every admissible 46-tuple contain at least two primes. Applying this to the explicitly verified admissible tuple $\mathcal{H}_{46}$ of diameter 216 gives $H_1 \leq 216$. □

10 What remains to be done

The recheck therefore leaves two concrete tasks.

1. **Analytic repair.** Supply a rigorous corrected version of Stadlmann's equidistribution criterion. The negative-ω_0 interval cannot be left as printed, and the other gaps previously identified in the Type I/IIc/III reductions must be closed.

2. **Finite-dimensional certificate.** Compute M_1, M_2 for a sufficiently strong symmetric basis on the *actual* restricted support (4), rationalize a candidate coefficient vector, and verify (11) exactly.

The important positive conclusion is that the candidate parameter set itself survives a careful arithmetic and combinatorial audit. In particular, the Type IIc partition for $\omega_0 \geq 0$ is not merely numerical: the one-coordinate argument above proves it uniformly for all allowed γ and ω_0.

References

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