Divergence in a Collatz-Type Recursion

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Abstract: The following recursion rule is applied to odd natural numbers n. $\sqrt{2} \cdot n + 4$ is downrounded (symbol $\lfloor ... \rfloor$), and then divided by 2 as often as possible. The outcome is the new n. We prove: For each odd $n(0)$ the resulting sequence $n(0), n(1), ...$ diverges to infinity. More detailled, after some short starting segment in each step t either $n(t+1) > n(t)$ or $n(t+2) > n(t)$. This gives an example for a Collatz-type recursion which does not end in finitely many limit cycles.

Key words: Collatz conjecture, $3x + 1$ problem, generalization, counter example.

1 Introduction

In 1937, Lothar Collatz analysed the following recursion scheme and formulated the conjecture that each odd starting value n converges to the limit cycle $1 - 4 - 2 - 1$ [2]:

$n \to 3n + 1$, and then halving, until an odd number results. Until now, no rigorous proof has been found. The variant with $3n - 1$ instead of $3n + 1$ results in a slightly more complicated structure: Each odd starting value n runs into one of the three limit cycles: $1 - 1$; $5 - 7 - 5$; $17 - 25 - 37 - 55 - 41 - 61 - 91 - 17$.

We introduced a generalized model: Given real parameters (x, y) with $1 < x < 2$ and $y > 0$, the recursion rule for odd number n is:

$x \cdot n + y$ is downrounded, and halving is done until an odd number is reached. This is the new n. The model for $x = \frac{3}{2}$ and $y = 0$ is equivalent to the $3n - 1$ problem mentioned above.

Our original conjecture was: For each pair (x, y) all odd starting values run into finitely many cycles. And we had the hope to be able to prove this conjecture for at least one pair (x, y). So far without success. Instead, we found a counter example, namely for $x = \sqrt{2}$ and $y = 4$.

2 The Theorem

Theorem: For the recursion $n \to \lfloor \sqrt{2} \cdot n + 4 \rfloor$ and division by 2, until an odd number is reached, each odd starting value $n(0)$ gives a sequence $(n(t))_{t=0,...}$ which runs to infinity.

In detail: After a short starting segment some step T is reached such that for all $t \geq T$ either $n(t + 1) > n(t)$ or $n(t + 2) > n(t)$. T satisfies $T \leq \log n(0)$ (log with base $\sqrt{8}$) and $n(t + 1) \leq n(t) \cdot \frac{\sqrt{2}+4}{4}$ for all $t \leq T$.

3 Some Data for Numerical Evidence

$n = 1$ gives the following sequence (shown until step $t = 100$). In each column two values are given: t and $n(t)$. Columns t with $n(t + 1) < n(t)$ are marked by an asterisk.

0, 1
1, 5
2, 11
3, 19 *
4, 15
5, 25
6, 39
7, 59
8, 87
9, 127
10, 183 *
11, 131
12, 189
13, 271
14, 387
15, 551
16, 783
17, 1111
18, 1575
19, 2231
20, 3159
21, 4471 *
22, 3163
23, 4477
24, 6335
25, 8963
26, 12679 *
27, 8967
28, 12685
29, 17943
30, 25379
31, 35895
32, 50767
33, 71799
34, 101543
35, 143607 *
36, 101547
37, 143613
38, 203103
39, 287235
40, 406215 *
41, 287239
42, 406221
43, 574487
44, 812451
45, 1148983
46, 1624911
47, 2297975
48, 3249831
49, 4595959
50, 6499671 *

51, 4595963
52, 6499677
53, 9191935
54, 12999363
55, 18383879
56, 25998735
57, 36767767 *
58, 25998739
59, 36767773
60, 51997487
61, 73535555
62, 103994983
63, 147071119
64, 207989975
65, 294142247 *
66, 207989979
67, 294142253
68, 415979967
69, 588284515
70, 831959943 *
71, 588284519
72, 831959949
73, 1176569047
74, 1663919907
75, 2353138103
76, 3327839823
77, 4706276215
78, 6655679655 *
79, 4706276219
80, 6655679661
81, 9412552447
82, 13311359331
83, 18825104903 *
84, 13311359335
85, 18825104909
86, 26622718679
87, 37650209827
88, 53245437367 *
89, 37650209831
90, 53245437373
91, 75300419671
92, 106490874755
93, 150600839351 *
94, 106490874759
95, 150600839357
96, 212981749527
97, 301201678723
98, 425963499063 *
99, 301201678727
100, 425963499069

For all t with an asterisk, we see:
1. There is only one halving
and
2. Column $n(t+1)$ does not give a halving.
Proof of these two observations is the kernel of our proof.
In a normal step (without asterisk) in the table there is no halving. So $n(t+1) = \lfloor \sqrt{2} \cdot n(t) + 4 \rfloor > \sqrt{2} \cdot n(t)$.
In the table, in a step with asterisk, $n(t+1) = \frac{\lfloor \sqrt{2} \cdot n(t)+4 \rfloor}{2} > \frac{n(t)}{\sqrt{2}}$.

4 Proof of the Theorem

Let $\alpha = \sqrt{2}$. For an odd integer n set
$k(n) := \lfloor \alpha n \rfloor$,
$R(n) := k(n) + 4$,
$e(n) := v_2(R(n))$, which is the number of factor 2 in the prime decomposition of $R(n)$.
Then the next odd number is given by $T(n) := \frac{R(n)}{2^{e(n)}}$.
Since α is irrational, $\alpha n \notin Z$ for all integers n, so if we write
$\alpha n = k(n) + x$ with $x \in (0, 1)$ then x is uniquely determined.

Lemma (transition of $\lfloor \alpha T(n) \rfloor$)
With $x \in (0, 1)$ as above two statements hold:
1. If $k(n)$ is odd (so $e(n) = 0$ and $T(n) = k(n) + 4$), then $\alpha \cdot T(n) = 2n - \alpha x + 4\alpha$ which gives $\lfloor \alpha T(n) \rfloor$ is either $2n + 4$ or $2n + 5$. (In particular, modulo 4 the term is 2 or 3, because n is odd.)
2. If $k(n) = 2 \pmod 4$ (so $e(n) = 1$ and $T(n) = \frac{k(n)+4}{2}$), then
$\alpha T(n) = n - \frac{\alpha x}{2} + 2\alpha$, $2.12... < 2\alpha - \frac{\alpha x}{2} < 2.82...$
hence
$\lfloor \alpha T(n) \rfloor = n + 2$ (which is odd).

Both statements follow by substituting $\alpha \cdot n = k(n) + x$ and using $\alpha^2 = 2$.
Now, for simplicity we look at the key invariant for the orbit starting at $n = 1$.
$I(n) := \lfloor \alpha n \rfloor \neq 0 \pmod 4$.
We claim $I(n_t)$ holds for every term n_t on the orbit starting at $n_0 = 1$. We prove this by induction.
Base step: $k(1) = \lfloor 1 \cdot \sqrt{2} \rfloor = 1 = 1 \text{ mod}(4)$, so $I(n_0)$ holds.
Induction step: Assume $I(n)$ holds for some odd n. Then $k(n) = 1$ or 2 or $3 \pmod 4$.
Case consideration:

- If $k(n)$ is odd (1 or 3 mod 4) then statement 1 of the Lemma gives $\lfloor \alpha T(n) \rfloor \in 2n + 4, 2n + 5$, hence unequal $0 \pmod 4$.
- If $k(n) = 2 \pmod 4$, then statement 2 of the Lemma gives $\lfloor \alpha T(n) \rfloor = n + 2$, which is odd, hence unequal $0 \pmod 4$.

In both cases $I(T(n))$ holds. By induction, $I(n_t)$ holds for all $t \geq 0$.

The consequences for halving are that at any step

$$e(n) = v_2(k(n)+4) = \begin{cases} 0, \text{ if } k(n) = 1 \text{ or } 3 \pmod 4, \\ 1, \text{ if } k(n) = 2 \pmod 4, \\ 2 \text{ or larger}, \text{ if } k(n) = 0 \pmod 4 \end{cases}$$

Since $I(n_t)$ holds for all t, the last case never occurs along the 1-orbit. Therefore:

1. At most one halving per step.

and

2. No consecutive halving steps. If a halving occurs at step t, then $k(n_t) = 2 \pmod 4$, and by statement 2 of the Lemma $\lfloor \alpha n_{t+1} \rfloor = n_t + 2$ is odd, so $e(n_{t+1}) = 0$.

This proves the two claims for the entire orbit starting at $n_0 = 1$.

Remark: The same argument works for any odd starting number n_0 with $\lfloor \alpha n_0 \rfloor \neq 0 \pmod 4$. The invariant prevents ever hitting 0 (mod 4), so multi-halving rounds can never occur.

It remains to explain what happens for starting values n with $\alpha n_0 = 0 \pmod 4$. Here, in each of the initial rounds $\alpha n + 4$ is downrounded and divided by 4 or a higher power of 2. As the numbers n_t never become negative, such a *meteor strike* has to stop after at most $\log n(0)$ steps (log with base $\sqrt{8}$). Then $\lfloor \alpha n_t \rfloor$ becomes $\neq 0 \pmod 4$ and the proof from above can be applied.

5 Discussion

1. We strongly believe that in the generalized model pairs (x, y) are seldom, for which there are not finitely many cycles, who catch all starting values. However, so far not a single pair (x, y) is known for which convergence to finitely many limit cycles has been proven.

2. Without theoretical proof our computer has shown long sequences (using double precision floating point (64 bit)) which seem to indicate that also in the following cases sequences run to infinity:

$x = \sqrt{2}$, $y = 6$, $n = 1$,

$x = \sqrt{2}$, $y = 24$, $n = 1$,

x=$\sqrt[3]{4}$, $y = 5.5$, $n = 5$.

3. For $x = \frac{4}{3}$ and $y = \frac{5}{2}$ we have a partial proof for divergence at $n = 5$.

4. Possibly, ideas of this proof can be applied also to variants of the Collatz problem as discussed in [1].

Acknowledgement: Thanks to Dietmar Wolz for helpful discussions how to use ChatGPT-5 in such proofs, Ulrich Tamm for technical support, and to the designers of ChatGPT-5.

References

[1] I. Althöfer, M. Hartisch, and T. Zipproth. Analysis of a Collatz Game and other variants of the $3n+1$ Problem. ACG 2023, Springer Lecture Notes in Computer Science 14528 (2024), 123 - 132.

[2] J. Lagarias. The $3x + 1$ problem and its generalizations. American Mathematical Monthly 1985 (92), 323.