Two Explicit Cylinder Proofs for a Three-Branch Stochastic Collatz Model

Almost-sure descent below 10^5 for the probability splits
(0.5, 0.4, 0.1) and $(0.50, 0.44, 0.06)$

ChatGPT 5.6 directed by Ingo Althöfer – ingo.althoefer@uni-jena,de

30 July 2026

Abstract

For a positive integer m, let $\mathrm{odd}(m)$ denote the odd part of m. We study a stochastic process on the positive odd integers in which one of

$$T_{+1}(n) = \mathrm{odd}(3n+1), \qquad T_{-1}(n) = \mathrm{odd}(3n-1), \qquad T_{+3}(n) = \mathrm{odd}(3n+3)$$

is chosen independently at each step. We give exact finite-cylinder Lyapunov certificates proving almost-sure descent below 10^5 for the two probability triples

$$(0.50, 0.40, 0.10) \quad \text{and} \quad (0.50, 0.44, 0.06).$$

The first proof is deliberately presented in a plain form. It needs only the sixteen odd residue classes modulo 32. The second proof uses 2048 odd residue classes modulo 4096 and is verified by exact integer arithmetic.

1 The stochastic model and the target

For every positive integer m, write

$$\mathrm{odd}(m) = \frac{m}{2^{v_2(m)}},$$

where $v_2(m)$ is the exponent of 2 in m. Thus $\mathrm{odd}(m)$ is always odd.

Starting from an odd integer $X_0 = n$, define a Markov chain by choosing one of T_{+1}, T_{-1}, T_{+3} independently at each step and setting

$$X_{t+1} = T_q(X_t).$$

We consider the two probability distributions

	T_{+1}	T_{-1}	T_{+3}
Case I	0.50	0.40	0.10
Case II	0.50	0.44	0.06.

Let

$$\tau = \inf\{t \geq 0 : X_t < 100000\}.$$

Because every X_t is odd, the event $\{\tau > t\}$ implies $X_t \geq 100001$.

The aim is to prove

$$\mathbb{P}_n(\tau < \infty) = 1$$

for every odd starting value $n > 100000$.

2 The finite-cylinder Lyapunov principle

A residue class modulo a power of two is often called a binary or 2-adic *cylinder*. Fix

$$M = 2^K$$

and assign a positive weight w_r to every odd residue $r \pmod M$. For some exponent $\alpha > 0$, put

$$V(n) = w_{n \bmod M}\, n^\alpha$$

for positive odd n.

The point of using cylinders is that a residue $r \pmod{2^K}$ determines a large part of the division by powers of two in $3n+q$. For an odd residue r and $q \in \{1, -1, 3\}$, set

$$y_q(r) \equiv 3r + q \pmod M, \qquad 0 \le y_q(r) < M.$$

If $y_q(r) \neq 0$, then

$$a_q(r) = v_2(y_q(r)) < K$$

is the exact value of $v_2(3n+q)$ for every $n \equiv r \pmod M$. After division by $2^{a_q(r)}$, the output residue is fixed modulo $2^{K-a_q(r)}$. Consequently, only $2^{a_q(r)}$ odd residues modulo M can occur.

If $y_q(r) = 0$, we only know that

$$v_2(3n+q) \ge K.$$

For an upper bound we may then allow every odd output residue and use the smallest possible division factor 2^K. This is conservative and therefore safe.

For all $n \ge 100001$ we have

$$\frac{3n+1}{n} \le \frac{300004}{100001}, \qquad \frac{3n-1}{n} < 3, \qquad \frac{3n+3}{n} \le \frac{300006}{100001}.$$

It is convenient to write

$$c_{+1} = \frac{300004}{100001}, \qquad c_{-1} = 3, \qquad c_{+3} = \frac{300006}{100001}.$$

For a fixed input cylinder r, let $\mathcal{S}_q(r)$ be the set of possible output cylinders described above. Then

$$\frac{V(T_q(n))}{V(n)} \le \frac{\max_{s \in \mathcal{S}_q(r)} w_s}{w_r} \left(\frac{c_q}{2^{a_q(r)}}\right)^\alpha, \qquad n \equiv r \pmod M,$$

where in the exceptional case $y_q(r) = 0$ we use $a_q(r) = K$ and all odd residues as $\mathcal{S}_q(r)$.

Lemma 1 (Cylinder contraction implies almost-sure descent)**.** *Suppose that, for some $0 < \rho < 1$, the cylinder weights satisfy*

$$\mathbb{E}[V(X_{t+1}) \mid X_t = n] \le \rho V(n) \tag{1}$$

for every odd $n \ge 100001$. Then every odd starting value $n > 100000$ reaches the region below 100000 with probability one.

Proof. Put

$$Z_t = V(X_t)\mathbf{1}_{\{\tau > t\}}.$$

On $\{\tau > t\}$, inequality (1) applies; after the chain has entered the target region, the killed variable is zero. Hence

$$\mathbb{E}[Z_{t+1} \mid \mathcal{F}_t] \le \rho Z_t$$

and therefore

$$\mathbb{E}[Z_t] \le \rho^t V(n).$$

Let $w_{\min} = \min_r w_r > 0$. On $\{\tau > t\}$ we have $X_t \ge 100001$, so

$$Z_t \ge w_{\min}\, 100001^{\alpha} \mathbf{1}_{\{\tau > t\}}.$$

Thus

$$\mathbb{P}_n(\tau > t) \le \frac{V(n)}{w_{\min} 100001^{\alpha}}\, \rho^t \longrightarrow 0.$$

It follows that $\mathbb{P}_n(\tau = \infty) = 0$. □

3 The plain sixteen-cylinder proof for $(0.50, 0.40, 0.10)$

This case has a particularly small certificate. We take

$$K = 5, \qquad M = 32, \qquad \alpha = \frac{1}{16}.$$

There are only sixteen odd residue classes modulo 32.

3.1 The weights

The following table gives exact integer weights W_r. We use

$$w_r = \frac{W_r}{10^{15}}.$$

In particular, every normalized weight is at least 1.

r	W_r	r	W_r
1	1073318014418258	17	1041796984172814
3	1029341385815588	19	1028154077897707
5	1022077403373602	21	1000000000000000
7	1031015602320025	23	1050800171836819
9	1046489827260072	25	1032888890360140
11	1026424044965650	27	1025017950735553
13	1032172211240146	29	1021418251463214
15	1040244855558753	31	1039163515378852

3.2 Exact upper bounds for the sixteenth roots

Set $S = 10^{18}$. For $q \in \{1, -1, 3\}$ and $1 \le a \le 5$, let $R_{q,a}$ be the least integer for which

$$R_{q,a}^{16}\, v_q 2^a \ge S^{16} u_q,$$

where $c_q = u_q / v_q$ in lowest terms. Therefore

$$\left(\frac{c_q}{2^a}\right)^{1/16} \le \frac{R_{q,a}}{S}.$$

The exact ceiling integers are as follows.

a	$R_{+1,a}$	$R_{-1,a}$	$R_{+3,a}$
1	1025665610144252763	1025665396466432482	1025666037497890129
2	982180753173840684	982180548555258971	982181162409085845
3	940539511478265840	940539315534840700	940539903363279184
4	900663721818221134	900663534182154388	900664097088595568
5	862478534819316063	862478355138402969	862478894179457770

Every number in this table is verified by the two exact integer inequalities

$$(R_{q,a}-1)^{16}v_q2^a < S^{16}u_q \le R_{q,a}^{16}v_q2^a.$$

No floating-point approximation enters the proof.

3.3 The sixteen row checks

For each odd residue $r \pmod{32}$, let

$$W_q^{\max}(r) = \max_{s\in\mathcal{S}_q(r)} W_s.$$

Because the probabilities are 5/10, 4/10, and 1/10, it is enough to check

$$5R_{+1,a_{+1}(r)}W_{+1}^{\max}(r) + 4R_{-1,a_{-1}(r)}W_{-1}^{\max}(r) + R_{+3,a_{+3}(r)}W_{+3}^{\max}(r) \le 10S\,\frac{9998}{10000}\,W_r. \quad (2)$$

There are only sixteen inequalities in (2), one for each odd residue modulo 32. The supplied self-contained verifier checks all of them with Python integers.

The largest row occurs at

$$r = 7.$$

Its exact ratio is

$$\frac{10307276588363010558174677057829510}{10310156023200250000000000000000000} = 0.9997207185972007\ldots,$$

which is smaller than

$$\rho_1 = \frac{9998}{10000} = 0.9998.$$

Thus (1) holds with $\rho = \rho_1$, and the lemma proves almost-sure descent below 100000.

Theorem 1. *For the probability split*

$$(\mathbb{P}(T_{+1}), \mathbb{P}(T_{-1}), \mathbb{P}(T_{+3})) = (0.50, 0.40, 0.10),$$

every odd starting value $n > 100000$ reaches a value below 100000 with probability one.

4 The 2048-cylinder proof for $(0.50, 0.44, 0.06)$

The stronger split

$$(\mathbb{P}(T_{+1}), \mathbb{P}(T_{-1}), \mathbb{P}(T_{+3})) = (0.50, 0.44, 0.06)$$

also has a compact exact certificate, although it is no longer convenient to print the complete weight table in the report.

We take

$$K = 12, \qquad M = 4096, \qquad \alpha = \frac{1}{32}.$$

There are 2048 odd residue classes. The certificate contains an integer weight W_r for each one, normalized by

$$w_r = \frac{W_r}{10^{15}}.$$

The weights satisfy

$$10^{15} \le W_r \le 1139727324861073.$$

As in the first proof, set $S = 10^{18}$ and compute exact upper ceilings for

$$\left(\frac{c_q}{2^a}\right)^{1/32}, \qquad q \in \{1, -1, 3\}, \quad 1 \le a \le 12.$$

For every odd residue $r \pmod{4096}$, the verifier then checks

$$50R_{+1,a_{+1}(r)}W^{\max}_{+1}(r) + 44R_{-1,a_{-1}(r)}W^{\max}_{-1}(r) + 6R_{+3,a_{+3}(r)}W^{\max}_{+3}(r) \le 100S\,\frac{99995}{100000}\,W_r. \quad (3)$$

All computations in (3), including the thirty-six root ceilings, use exact integer arithmetic.

The worst residue is

$$r = 245 \pmod{4096}.$$

Its exact row ratio is

$$\frac{1048510812612784956454501920827881 66}{1048588069560561000000000000000000000} = 0.9999263228811973\ldots,$$

and therefore it is below

$$\rho_2 = \frac{99995}{100000} = 0.99995.$$

The cylinder contraction lemma applies again.

Theorem 2. *For the probability split*

$$(\mathbb{P}(T_{+1}), \mathbb{P}(T_{-1}), \mathbb{P}(T_{+3})) = (0.50, 0.44, 0.06),$$

every odd starting value $n > 100000$ *reaches a value below* 100000 *with probability one.*

5 Correctness audit and reproducibility

Both proofs were rechecked in several independent ways.

1. The exact certificate programs were rerun from scratch. They recompute all root ceilings by integer binary search and verify every cylinder inequality using arbitrary-precision integers.
2. The construction of the possible successor cylinders was audited independently. If $3r+q \not\equiv 0 \pmod{2^K}$, the computed valuation is exact and the output classes are precisely those congruent modulo 2^{K-a}. If $3r + q \equiv 0 \pmod{2^K}$, using $a = K$ and the global maximum weight is a valid conservative bound.
3. As a supplementary implementation check, the actual one-step drift was computed directly for every odd

   $$100001 \le n \le 10000001,$$

   that is, for 4,950,001 consecutive odd starting states in each model. A further 250,000 randomly generated large odd states were checked for each probability split. No violation was found. These numerical checks are not needed for the proof; the exact finite-cylinder inequalities are decisive.

The reproducibility package contains:

- a self-contained verifier for the sixteen-cylinder proof;
- a portable verifier and binary weight table for the 2048-cylinder proof;
- a short README file with execution instructions.

The exact scripts and weight file are included in the accompanying certificate package.

6 Conclusion

The split $(0.50, 0.40, 0.10)$ admits a very plain proof: sixteen binary cylinders, sixteen weights, and sixteen exact inequalities. Increasing the $3n - 1$ probability from 0.40 to 0.44 while reducing the $3n + 3$ probability from 0.10 to 0.06 still leaves a manageable certificate, now with 2048 cylinders. Both proofs establish almost-sure entrance into the region below 10^5 for every odd starting value above 10^5.