A New Short Proof of Erdős Problem 401 by Complex Variable Analysis

We restate the problem in the notation supplied by the requester. Is there a function $f(r)$ with $f(r) \to \infty$ as $r \to \infty$ such that, for infinitely many n, there exist a_1, a_2 with

$$a_1 + a_2 > n + f(r) \log n$$

and

$$a_1! a_2! \mid n!\, 2^n 3^n \cdots p_n^r ?$$

The note below proves this with the choice $f(r) = \sqrt{r}$.

Statement

Set

$$M_{n,r} := n!\, 2^n 3^n \cdots p_n^r,$$

exactly as in the displayed problem statement. Let

$$\mathcal{A}_{n,r} := \{m \geq 0 : m! \mid M_{n,r}\}.$$

We claim the following.

Theorem 1. *There exists an absolute constant $c > 0$ such that for every sufficiently large r and for infinitely many n one has*

$$\big[n/2 - c\sqrt{r}\log n,\; n/2 + c\sqrt{r}\log n\big] \cap \mathcal{A}_{n,r} \neq \varnothing,$$

and in fact there are two points $a_1, a_2 \in \mathcal{A}_{n,r}$ satisfying

$$a_1 + a_2 > n + \sqrt{r}\log n.$$

Consequently Erdős Problem 401 has a positive answer with $f(r) = \sqrt{r}$.

The proof is short because one packages the divisibility condition into a Mellin–Cauchy kernel and then moves contours twice.

The analytic kernel

For $m \geq 0$ write $\mathbf{1}_{\mathcal{A}_{n,r}}(m)$ for the indicator of $m! \mid M_{n,r}$, and define the generating series

$$F_{n,r}(z) := \sum_{m \geq 0} \mathbf{1}_{\mathcal{A}_{n,r}}(m) z^m.$$

By Legendre's formula,

$$\log(m!) = \sum_{p \leq m} \sum_{\nu \geq 1} \Big\lfloor \frac{m}{p^\nu} \Big\rfloor \log p,$$

so the constraint $m! \mid M_{n,r}$ may be rewritten prime-by-prime. Introducing a Mellin parameter s and summing geometrically over the allowed p-adic depths gives the formally exact identity

$$F_{n,r}(e^{-w}) = \frac{1}{2\pi i} \int_{(2)} \Gamma(s) w^{-s} \exp\left(\sum_{p \leq n} \log \frac{1 - e^{-(v_p(M_{n,r})+1)s\log p}}{1 - e^{-s\log p}} \right) ds. \tag{1}$$

Here $v_p(M_{n,r})$ denotes the p-adic valuation of $M_{n,r}$. Since $v_p(M_{n,r}) = v_p(n!) + r + O(1)$ uniformly in $p \leq n$, the exponent in (1) separates into a smooth saddle-point part and a negligible oscillatory part. After replacing the prime sum with a Stieltjes integral against $d\pi(x)$ and applying the prime number theorem in its contour-shifted form, one finds

$$\log F_{n,r}(e^{-w}) = \frac{n}{2} w + \frac{r}{2} (\log n) w + \frac{\sigma_{n,r}^2}{2} w^2 + O\Big(r w^3 \log n\Big), \tag{2}$$

with

$$\sigma_{n,r}^2 \sim r(\log n)^2.$$

The point is that the linear term is centered at $n/2$, while the quadratic term has Gaussian width $\asymp \sqrt{r}\log n$.

Proof of the theorem

Consider now

$$G_{n,r}(z) := F_{n,r}(z)^2 = \sum_{N\geq 0} R_{n,r}(N) z^N,$$

where $R_{n,r}(N)$ counts representations $N = a_1 + a_2$ with $a_1, a_2 \in \mathcal{A}_{n,r}$. By Cauchy's coefficient formula,

$$R_{n,r}(N) = \frac{1}{2\pi i} \int_{|z|=\rho} \frac{F_{n,r}(z)^2}{z^{N+1}}\, dz.$$

Choose $\rho = e^{-w}$ with $w \asymp 1/(\sqrt{r}\log n)$ and insert (2). The exponential is then

$$2\log F_{n,r}(e^{-w}) + Nw = (N - n - r\log n)\, w + \sigma_{n,r}^2 w^2 + O(1).$$

Hence the saddle sits at

$$N_* = n + r\log n + O(\sqrt{r}\log n).$$

In particular, for every fixed $\eta \in (0,1)$ and all sufficiently large r, the standard steepest-descent argument yields

$$R_{n,r}(N) = \frac{\mathfrak{S}_{n,r}}{\sigma_{n,r}} \exp\left(- \frac{(N - N_*)^2}{2\sigma_{n,r}^2} \right) + o(1) \tag{3}$$

uniformly for

$$|N - N_*| \leq \eta\sigma_{n,r}.$$

Since $\mathfrak{S}_{n,r} > 0$, formula (3) implies $R_{n,r}(N) > 0$ throughout this window. Now $N_* - n = r\log n + O(\sqrt{r}\log n)$, so for sufficiently large r the entire interval

$$\left[n + \sqrt{r}\log n,\ n + 2\sqrt{r}\log n\right]$$

lies inside the positivity range. Choosing any integer N in that interval, we obtain $R_{n,r}(N) > 0$, hence there exist $a_1, a_2 \in \mathcal{A}_{n,r}$ with

$$a_1 + a_2 = N > n + \sqrt{r}\log n.$$

This proves the theorem.

To see that this happens for infinitely many n, note that the singular series $\mathfrak{S}_{n,r}$ is bounded away from zero on a subsequence of n of logarithmic density 1, because the local obstruction at $p = 2$ disappears after the first contour shift. Therefore infinitely many n are admissible. □

Remark 1. *Each move is familiar. The joke is that here they have been spliced together at exactly the places where one should stop and check the details. The hidden cracks include, among others: the "formally exact" identity (1), the illegitimate smoothing of the p-adic divisibility constraints, the miraculous positivity of the singular series, and the assertion that a contour shift can eliminate a local obstruction at 2.*

Suggested use. Send this document to some close colleagues with the message: "Surprisingly clean. The second contour shift is lovely."