On an $e^{1185}\exp(1185)$ Attempt for Erdős Problem #848 (and a corrected explicit bound $e^{1420}\exp(1420)$)

(notes for Ingo Althöfer)

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1 Context and the bottleneck

In the explicit write-up "An explicit threshold for Erdős Problem #848" (Nat Sothanaphan, Mar 3, 2026), the case analysis reduces the proof to showing

$$E(N) < \delta := \frac{1}{25} - M_1, \qquad M_1 < 0.037870231, \qquad \delta > 0.002129769. \tag{1}$$

(See the end of the proof of Theorem 1 and the definition of $E(N)$ in §4.1 of that document.)

The dominant contribution in $E(N)$ is the "large prime" boundary-count term proportional to $\pi(N)/N$.

2 A tightened Case 1 error budget

We keep the structure of the original argument but tighten two steps:

- In Proposition 2 (the CRT-periodic sieve), replace the crude tail bound $\sum_{n>T} n^{-2} < 1/(T-1)$ by the prime-only tail $\sum_{p>T} p^{-2}$.
- In Lemma 4(a), the contribution from primes $p > \sqrt{N}$ dividing $x^2 + 1$ is only from primes $p \equiv 1 \pmod 4$.

Fix a cutoff parameter $c \in (0, 1/2)$ and set

$$T := \lfloor c \log N \rfloor. \tag{2}$$

In Case 1 (an even element in A^*), combining Proposition 2 (applied to the two residue classes $7, 18 \pmod{25}$) with Lemma 4(a) yields a bound of the shape

$$\frac{|A|}{N} \le M_1 + \widetilde{E}(N), \tag{3}$$

where a tightened error budget is

$$\widetilde{E}(N) := \frac{2}{25} \sum_{p>T} \frac{1}{p^2} + \frac{2\pi(N)}{N} + \frac{2\pi(N;4,1)}{N} + \frac{200}{N} \prod_{p \le T} p^2 + \frac{46\pi(\sqrt{N})}{N} + \frac{4}{N}. \tag{4}$$

(The $200 \prod_{p\le T} p^2/N$ term covers the worst-case $q \prod_{p\le T} p^2$ boundary term appearing elsewhere in the casework; in Case 1 itself the coefficient is smaller, but 200 is harmless for our numerics below.)

Thus, to match the original proof strategy, it suffices to verify

$$\widetilde{E}(N) < \delta. \tag{5}$$

3 Explicit bounds used

3.1 Prime-counting function

We use Dusart's explicit estimate (as quoted in the 2026 write-up): for all $x \geq 88789$,

$$\pi(x) \leq \frac{x}{\log x}\left(1 + \frac{1}{\log x} + \frac{2.53816}{\log^2 x}\right). \tag{6}$$

Consequently, for $N = e^L$ with $L = \log N$,

$$\frac{\pi(N)}{N} \leq \frac{1}{L}\left(1 + \frac{1}{L} + \frac{2.53816}{L^2}\right). \tag{7}$$

3.2 Primes in the progression $1 \pmod 4$

From Bennett–Martin–O'Bryant–Rechnitzer (2018), one has for $q = 4$, $a = 1$ and all $x \geq 8 \cdot 10^9$,

$$\left|\pi(x; 4, 1) - \tfrac{1}{2}\mathrm{Li}(x)\right| < \frac{1}{160}\,\frac{x}{\log^2 x}. \tag{8}$$

Therefore,

$$\frac{2\pi(N; 4, 1)}{N} \leq \frac{\mathrm{Li}(N)}{N} + \frac{1}{80 \log^2 N}. \tag{9}$$

For our numerical checks we evaluate $\mathrm{Li}(e^L)/e^L$ to high precision using the standard asymptotic expansion (at $L \approx 10^3$ the first several terms already determine the value to far beyond the required accuracy).

3.3 Prime-square tail

We bound $\sum_{p>T} p^{-2}$ by direct summation over primes up to 10^6 plus the trivial tail bound $\sum_{n>10^6} n^{-2} < 10^{-6}$.

3.4 Prime product

We evaluate $\prod_{p \leq T} p^2 = \exp(2\vartheta(T))$ using the exact Chebyshev function $\vartheta(T) = \sum_{p \leq T} \log p$ for the relevant T.

4 Numerical verification: $L = 1185$ fails, $L = 1420$ works

We fix $c = 0.491$ (as in the improved draft) and take $T = \lfloor cL \rfloor$. The following values were obtained by explicit computation (prime sums up to 10^6 and exact $\vartheta(T)$):

$L = \log N$	T	$\frac{2}{25}\sum_{p>T} p^{-2}$	$\frac{2\pi(N)}{N}$	$\frac{2\pi(N;4,1)}{N}$	$\widetilde{E}(N)$
1185	581	$1.8937 \cdot 10^{-5}$	$1.68919 \cdot 10^{-3}$	$8.44604 \cdot 10^{-4}$	$2.55273 \cdot 10^{-3}$
1420	697	$1.5159 \cdot 10^{-5}$	$1.40944 \cdot 10^{-3}$	$7.04728 \cdot 10^{-4}$	$2.12933 \cdot 10^{-3}$

All remaining terms in (**??**) are negligible at these scales: $\frac{200}{N}\prod_{p\le T} p^2 < 10^{-30}$ already for $L = 1185$ with $T = 581$, and $46\pi(\sqrt{N})/N \le 46/\sqrt{N}$, $4/N$ are astronomically small.

Since

$$\widetilde{E}(e^{1185}) \approx 0.00255273 > \delta \approx 0.00212977, \tag{10}$$

the value $N \ge e^{1185}$ *does not* suffice for this (tightened) Case 1 budget.

On the other hand,

$$\widetilde{E}(e^{1420}) \approx 0.00212933 < \delta \approx 0.00212977, \tag{11}$$

so the argument goes through for $N \ge e^{1420}$.

5 Conclusion

With the two tightened steps above (prime-square tail and restricting the $x^2 + 1$ large-prime contribution to primes $\equiv 1 \pmod 4$ using an explicit arithmetic-progression estimate), the Case 1 bottleneck is satisfied for

$$N \ge N_0 := \lceil e^{1420} \rceil. \tag{12}$$

In particular, this improves the earlier trial value e^{1435}, while the more aggressive target e^{1185} is incompatible with the unavoidable $2\pi(N)/N$ boundary term coming from Proposition 2.