Five powers of 2 suffice for Linnik–Goldbach

A conditional result from a fourth-moment input

Astra 6.0, prompted by Ingo Althoefer.

Revised version: 21 October 2026

Abstract

Five powers of 2 suffice, conditionally, in the Linnik–Goldbach problem: every sufficiently large even integer is the sum of two primes and five powers of 2. We obtain this conclusion assuming a fixed bound $\Re\rho \le \Theta < 11/12$ for every nontrivial zero of every primitive Dirichlet L-function, and taking the cited fourth-moment and major-arc estimates as inputs. Thus the five-power conclusion, previously obtained in the supplied paper under GRH, follows under this weaker family-wide zero hypothesis. Our contribution is to transfer the exceptional-set estimate to the required arcs, remove smoothing, and certify the resulting numerical inequality exactly. The proof also gives a weighted representation count of at least $NL^5/200$. The external analytic theorems and the zero hypothesis are not independently proved in this note.

1 Introduction and conditional main result

1.1 Five powers of 2 are enough

The main conclusion of this note is the following conditional statement:

> **Every sufficiently large even integer can be written as the sum of two primes and five powers of 2, provided the stated family-wide zero hypothesis and the imported analytic estimates hold.**

In symbols, for every sufficiently large even N there are primes p_1, p_2 and positive integers $\nu_1, \dots, \nu_5$ such that

$$N = p_1 + p_2 + 2^{\nu_1} + 2^{\nu_2} + 2^{\nu_3} + 2^{\nu_4} + 2^{\nu_5}.$$

The powers may repeat. The zero hypothesis requires one fixed $1/2 \le \Theta < 11/12$ such that $\Re\rho \le \Theta$ for every nontrivial zero of every primitive Dirichlet L-function. For example, a family-wide bound $\Theta = 7/8$ would meet this requirement. We use Theorem 1 and Proposition 4.1 of [2] as stated, together with the inputs from [1] specified below.

The supplied paper [1] proves a six-power result unconditionally and a five-power result under GRH. The advance established here is that *the five-power conclusion follows under the weaker zero hypothesis just described*, using the cited analytic inputs. The missing step is the passage from a smoothed fourth-moment bound to an exceptional-set estimate for the sharply truncated prime sum on the original arcs. Sections 3 and 4 prove that passage, and Sections 5 and 6 complete the reflection argument with exact numerical certificates. Theorem 1.1 gives the quantitative form.

1.2 Notation and external inputs

All logarithms are natural unless a base is indicated, and $\mathrm{e}(t) = \exp(2\pi i t)$. Let N tend to infinity through even integers and put

$$L = \left\lfloor \log_2 N - \sqrt{\log_2 N} \right\rfloor, \qquad G(\alpha) = \sum_{\nu=1}^{L} \mathrm{e}(2^\nu \alpha), \qquad z_L(\alpha) = G(\alpha)/L.$$

Thus $L \sim \log N / \log 2$ and $2^L = o(N)$. On $\mathbb{R}/\mathbb{Z}$ define

$$\mathfrak{M}(P) = \bigcup_{q \le P} \bigcup_{\substack{a \bmod q \\ (a,q)=1}} \left\{ \alpha : \left\| \alpha - \frac{a}{q} \right\| \le \frac{P}{qN} \right\}, \qquad \mathfrak{m}(P) = \mathbb{R}/\mathbb{Z} \setminus \mathfrak{M}(P). \tag{1.1}$$

These sets are nested in P; their individual arcs are disjoint when $2P^2 < N$. Write

$$S^\sharp(\alpha) = \sum_{p \le N} \log p \, \mathrm{e}(p\alpha), \qquad S_w(\alpha) = \sum_{p} \log p \, w(p/N) \mathrm{e}(p\alpha).$$

An admissible w is fixed, real, in $C_c^\infty((0,1))$, lies between zero and one, equals one on $[1/8, 7/8]$, and is Gevrey of order two in logarithmic coordinates. Constants may depend on this fixed weight.

For nonzero even h, set

$$\mathfrak{S}(h) = 2 \prod_{p>2} \left(1 - \frac{1}{(p-1)^2}\right) \prod_{\substack{p \mid h \\ p>2}} \frac{p-1}{p-2}, \qquad I_w(h) = \sum_{n \in \mathbb{Z}} w(n/N) w((n+h)/N).$$

The new analytic inputs are precisely the following statements.

(A1) For some fixed $\kappa_* \in (4/9, 9/20)$ and every fixed admissible w,

$$\int_{\mathfrak{m}(N^{\kappa_*})} |S_w(\alpha)|^4 \, d\alpha \ll_w N^{5/2} (\log N)^{100}. \tag{1.2}$$

(A2) For $\kappa = 11/25$ and $\kappa = \kappa_*$, and every such w, there is $c_\kappa > 0$ such that, uniformly for nonzero even $|h| \le N/2$,

$$\int_{\mathfrak{M}(N^{\kappa})} |S_w(\alpha)|^2 \mathrm{e}(-h\alpha) \, d\alpha = \mathfrak{S}(h) I_w(h) + O_w(N^{1-c_\kappa}). \tag{1.3}$$

Negative shifts follow from symmetry. No assertion at $h = 0$ is included in (1.3).

Li–Liu [2, Theorem 1 and Proposition 4.1] state these inputs under

$$\Re\rho \le \Theta \quad \text{for every nontrivial zero of every primitive Dirichlet } L\text{-function}, \qquad \tfrac{1}{2} \le \Theta < \tfrac{11}{12}. \tag{Q_Θ}$$

This is a hypothesis on the entire primitive family, including conductor one. A statement about ζ alone does not supply it.

Theorem 1.1 (Conditional five-power theorem)**.** *Assume* (A1) *and* (A2)*, together with the singular-series and major-arc inputs from [1] recorded in Section 5. Then every sufficiently large even integer is the sum of two primes and five powers of* 2*. More precisely, there are fixed $\kappa_0 > 0$ and N_0 such that every even $N \ge N_0$ satisfies*

$$\sum_{\substack{N = p_1 + p_2 + 2^{\nu_1} + \cdots + 2^{\nu_5} \\ N^{1-\kappa_0} < p_1, p_2 \le N, \quad 1 \le \nu_1, \ldots, \nu_5 \le L}} (\log p_1)(\log p_2) \;\ge\; \frac{1}{200} N L^5. \tag{1.4}$$

Primes and powers are ordered, and repeated powers are allowed.

Using the cited analytic results as stated, (Q_Θ) therefore suffices for the conclusion. We make no independent assertion here that (Q_Θ) has been proved, and no four-power conclusion is obtained.

2 The dyadic exceptional set

Put $\lambda = 25/32$, $r = \lambda^2/2 = 625/2048$, and

$$\alpha_- = \frac{r - \sqrt{r^2+4r}}{2}, \qquad \beta = \frac{r + \sqrt{r^2+4r}}{2}.$$

Define

$$\mathcal{D} = \left\{ x + iy : x \le \beta, \quad x^2 + y^2 \le \frac{r}{1 + r - x} \right\}.$$

Let $\mathcal{E} = \{\alpha \in \mathbb{R}/\mathbb{Z} : z_L(\alpha) \notin \mathcal{D}\}$. The dyadic large-deviation estimate in [1, Proposition on the exceptional set] yields

$$|\mathcal{E}| \ll e^{-(347/1000)L} = N^{-\gamma + o(1)}, \qquad \gamma = \frac{347}{1000 \log 2} > \frac{1}{2}. \tag{2.1}$$

Indeed, for

$$I(t) = \frac{(1+t)\log(1+t) + (1-t)\log(1-t)}{2} = \sum_{j \ge 1} \frac{t^{2j}}{2j(2j-1)},$$

the first eight terms at $t = 25/32$ already exceed $347/1000$. The bound $\log 2 < 347/500$ proves the second inequality in (2.1). For an entirely rational verification, use the first six terms in $\log 2 = 2\sum_{j \ge 0} 3^{-2j-1}/(2j+1)$ and bound its remaining tail by $2 \cdot 3^{-13}/(13(1 - 1/9))$.

We also need a lower bound for $|G|$ on $\mathcal{E}$, rather than just its measure. Since $\beta > 2/5$ and

$$\left(1 + r + \frac{2}{5}\right)\frac{4}{25} = 0.272828125 < r,$$

we have $\{|z| \le 2/5\} \subset \mathcal{D}$. Consequently

$$\alpha \in \mathcal{E} \quad \Longrightarrow \quad |G(\alpha)| > \frac{2}{5}L. \tag{2.2}$$

All terminating decimals in this note denote exact rational numbers.

For later use, let $r_{i,j}(h)$ count ordered tuples of exponents in $[1, L]$ with $h = \sum_{s=1}^{i} 2^{\nu_s} - \sum_{t=1}^{j} 2^{\mu_t}$. In particular,

$$\sum_h r_{2,2}(h) = L^4, \qquad r_{2,2}(0) = 2L^2 - L, \qquad r_{2,2}(h) \ne 0 \Longrightarrow |h| \le 2^{L+1}. \tag{2.3}$$

The zero count follows from uniqueness of an unordered sum of two powers of two: distinct exponents give two orderings and equal exponents give one. The singular-series calculation in [1] supplies

$$\sum_{h \ne 0} r_{2,2}(h)\mathfrak{S}(h) = 2L^4(1 + A_{22} + o(1)) \ll L^4, \qquad A_{22} < 0.0555. \tag{2.4}$$

3 Transferring the exceptional-set estimate between arcs

Proposition 3.1. *Under* (A1) *and* (A2)*, for each fixed admissible weight w,*

$$\int_{\mathcal{E} \cap \mathfrak{m}(N^{11/25})} |S_w(\alpha)|^2 \, d\alpha = o_w(N). \tag{3.1}$$

Proof. Set $P_- = N^{11/25}$ and $P_+ = N^{\kappa_*}$. Cauchy's inequality, (2.1), and (1.2) give

$$\int_{\mathcal{E}\cap\mathfrak{m}(P_+)} |S_w|^2 \le |\mathcal{E}|^{1/2} \left(\int_{\mathfrak{m}(P_+)} |S_w|^4 \right)^{1/2} \ll_w N^{5/4-\gamma/2+o(1)} (\log N)^{50} = o_w(N), \tag{3.2}$$

since $5/4 - \gamma/2 < 1$. Notice that (3.2) by itself does not cover $\mathfrak{m}(P_-)$.

Let $\mathcal{A} = \mathfrak{M}(P_+) \setminus \mathfrak{M}(P_-)$. For a cutoff P, write $R_{P,w}(h) = \int_{\mathfrak{M}(P)} |S_w|^2 \mathrm{e}(-h\alpha)\, d\alpha$. Both major-arc sets are invariant under $\alpha \mapsto -\alpha$, so these coefficients are real and even. Expanding $|G|^4$ gives the exact identity

$$\int_{\mathcal{A}} |S_w|^2 |G|^4 = \sum_h r_{2,2}(h)(R_{P_+,w}(h) - R_{P_-,w}(h)). \tag{3.3}$$

For all nonzero shifts occurring here, (1.3) applies for large N. Its main term is identical at the two cutoffs and cancels. With $c = \min(c_{11/25}, c_{\kappa_*}) > 0$, their total contribution in absolute value is $O_w(N^{1-c}L^4)$.

For the zero shift, use only Parseval and $0 \le w \le 1$:

$$0 \le R_{P_+,w}(0) - R_{P_-,w}(0) \le \int_{\mathbb{R}/\mathbb{Z}} |S_w|^2 \ll N \log N.$$

It follows from (2.3) and (3.3) that

$$\int_{\mathcal{A}} |S_w|^2 |G|^4 \ll_w N^{1-c}L^4 + NL^2 \log N.$$

Now (2.2) implies

$$\int_{\mathcal{E}\cap\mathcal{A}} |S_w|^2 \ll_w N^{1-c} + \frac{N \log N}{L^2} = o_w(N).$$

Since $\mathfrak{m}(P_-) = \mathfrak{m}(P_+) \cup \mathcal{A}$, up to boundaries of measure zero, this and (3.2) prove the result. □

4 Removing smoothing without losing a logarithm

We use the standard interval version of the upper-bound sieve for two linear forms: for an interval J of length $Y \ge 3$ and nonzero even $|h| \le Y$,

$$\#\{n \in J : n, n+h \text{ are positive primes}\} \ll \frac{Y\mathfrak{S}(h)}{(\log Y)^2}, \tag{4.1}$$

with an absolute constant, uniform in the position of J and in h. Only this coarse bound, not a sharp sieve constant, is needed here. To see the uniformity, sieve $n(n+h)$ on J. The number of roots modulo a prime is one if $p \mid h$ and two otherwise; for squarefree d the residue-class error is $O(2^{\omega(d)})$, independent of the translate of J. The usual Selberg upper-bound argument, with sieving level a small fixed power of Y, gives (4.1); pairs containing a prime below that level contribute a smaller error. The local factors are exactly those of $\mathfrak{S}(h)$. This is the same sieve calculation as [3, Section 2.3], with the interval length replacing the ambient cutoff and $n(n+h)$ replacing $n(N-n)$.

Proposition 4.1. *Under the assumptions of Proposition 3.1,*

$$\int_{\mathcal{E}\cap\mathfrak{m}(N^{11/25})} |S^\sharp(\alpha)|^2\, d\alpha = o(N). \tag{4.2}$$

The same conclusion holds for $S(\alpha) = \sum_{N^{1-\kappa_0} < p \le N} \log p\, \mathrm{e}(p\alpha)$ *with any fixed* $\kappa_0 > 0$.

Proof. Fix $0 < \delta < 1/8$. Choose an admissible w_δ that equals one throughout $[\delta, 1-\delta]$. Smooth Gevrey cutoffs with these properties exist for every fixed δ; their derivative bounds need not be uniform as $\delta \downarrow 0$. Put

$$D_\delta = S^\sharp - S_{w_\delta} = \sum_{p \le N} d_p \mathrm{e}(p\alpha), \qquad d_p = (\log p)(1 - w_\delta(p/N)).$$

Then $0 \le d_p \le \log N$, and its support is contained in the two intervals $(0, \delta N)$ and $((1-\delta)N, N]$. Let $C_\delta(h) = \sum_{p-p'=h} d_p d_{p'}$. For $0 < |h| \le 2^{L+1}$, eventually $|h| < \delta N$, and a pair cannot have its two primes in opposite endpoint intervals. Applying (4.1) on each interval gives

$$C_\delta(h) \ll \frac{\delta N (\log N)^2}{(\log(\delta N))^2} \mathfrak{S}(h) \ll \delta N \mathfrak{S}(h). \tag{4.3}$$

The final implied constant is independent of δ, once N is sufficiently large depending on δ: it suffices that $\log(\delta N) \ge \frac{1}{2} \log N$. At zero, the bound $C_\delta(0) \ll N \log N$ suffices. Expanding on the full circle and using (2.3) and (2.4), we obtain

$$\begin{aligned} \int_{\mathbb{R}/\mathbb{Z}} |D_\delta|^2 |G|^4 &= \sum_h r_{2,2}(h) C_\delta(h) \\ &\ll \delta N L^4 + N L^2 \log N. \end{aligned}$$

All terms are nonnegative, and the implied constant can be chosen independently of δ. By (2.2),

$$\int_{\mathcal{E}} |D_\delta|^2 \ll \delta N + \frac{N \log N}{L^2}. \tag{4.4}$$

The elementary inequality $|u+v|^2 \le 2|u|^2 + 2|v|^2$ and Proposition 3.1 now imply, for every fixed δ,

$$\limsup_{N \to \infty} \frac{1}{N} \int_{\mathcal{E} \cap \mathfrak{m}(N^{11/25})} |S^\sharp|^2 \le C\delta,$$

where C is absolute. Letting $\delta \downarrow 0$ proves (4.2). This is an iterated limit: we do not substitute a weight depending on N into (A1) or (A2).

Finally, Parseval gives

$$\int_{\mathbb{R}/\mathbb{Z}} \left| \sum_{p \le N^{1-\kappa_0}} \log p \, \mathrm{e}(p\alpha) \right|^2 d\alpha \ll N^{1-\kappa_0} \log N = o(N).$$

A second application of $|u+v|^2 \le 2|u|^2 + 2|v|^2$ proves the assertion for S. □

5 The five-power criterion on the original arcs

Here we record exactly the remaining inputs from [1] and derive the criterion. Fix a sufficiently small error parameter $\eta > 0$, then choose $\kappa_0 > 0$ sufficiently small in terms of η. The original major-arc construction gives a cutoff

$$N^{11/25} \le P \le N^{4/9 - 1/450}.$$

Write $\mathfrak{M} = \mathfrak{M}(P)$, $\mathfrak{m} = \mathfrak{m}(P)$ and $J = \int_{\mathfrak{M}} |S|^2$. Because $\mathfrak{m} \subseteq \mathfrak{m}(N^{11/25})$, Proposition 4.1 gives

$$\int_{\mathfrak{m} \cap \mathcal{E}} |S|^2 = o(N). \tag{5.1}$$

5.1 Singular-series and moment inputs

For odd squarefree $q > 1$, let $\ell_q = \mathrm{ord}_q(2)$ and

$$z_q(a) = \frac{1}{\ell_q}\sum_{t=1}^{\ell_q} \mathrm{e}(a2^t/q), \qquad A_{ij}(q) = \frac{1}{\varphi(q)^2}\sum_{\substack{a \bmod q\\(a,q)=1}} z_q(a)^i\overline{z_q(a)^j}, \quad A = \sum_q A(q).$$

The matrix A is real symmetric; it and every subseries are positive semidefinite. The singular-series estimates and mixed-major-arc lemma of [1] give, for $1 \le i, j \le 3$,

$$\sum_{h\neq 0} r_{i,j}(h)\mathfrak{S}(h) = 2L^{i+j}(1 + A_{ij} + o(1)), \tag{5.2}$$

$$\sum_{h\neq 0} r_{i,j}(h)\int_{\mathfrak{M}} |S|^2\mathrm{e}(-h\alpha)\,d\alpha = 2NL^{i+j}(1 + A_{ij} + O(\eta) + o(1)). \tag{5.3}$$

Also $r_{1,1}(0) = L$ and $r_{i,j}(0) \ll L^{i+j-2}$ when $i + j \ge 3$. The numerical diagonal inputs are

$$A_{11} < 0.279, \qquad A_{22} < 0.0555, \qquad A_{33} < 0.0136. \tag{5.4}$$

Using the prime-pair sieve bound with $C_1 = 6.7814$ as in [1], the full-circle base moment minus its major-arc part satisfies

$$\int_{\mathfrak{m}} |SG|^2 \le 2NL^2\left(\frac{C_1 - 2}{2}(1 + A_{11}) + \frac{\log 2}{2} + O(\eta) + o(1)\right) - LJ. \tag{5.5}$$

For completeness, the full-circle upper bound here is $NL^2(\log 2 + C_1(1 + A_{11}) + o(1))$, while the major-arc expression is $LJ + 2NL^2(1 + A_{11} + O(\eta) + o(1))$.

Only the following lower bound for J will be used:

$$J \ge (1 - o(1))N\log P, \qquad \frac{J}{2NL} \ge d + o(1), \qquad d = \frac{11\log 2}{50}. \tag{5.6}$$

Here is a direct justification. Since all primes in S exceed $q \le P$, the Ramanujan-sum identity and Cauchy's inequality imply

$$\sum_{\substack{a \bmod q\\(a,q)=1}} |S(a/q + u)|^2 \ge \frac{\mu(q)^2}{\varphi(q)}|S(u)|^2.$$

For fixed $T \ge 1$, sum over $q \le P/T$ and integrate over $|u| \le T/N$. The sum $\sum_{q\le P/T}\mu(q)^2/\varphi(q) = \log P + O_T(1)$, and the prime number theorem gives

$$\frac{1}{N}\int_{-T/N}^{T/N} |S(u)|^2\,du = \int_{-T}^{T}\left|\int_0^1 \mathrm{e}(vt)\,dt\right|^2 dv + o_T(1).$$

The last integral tends to one as $T \to \infty$, proving (5.6). *Audit correction:* the corresponding lemma in the supplied source states an asymptotic equality; its displayed proof establishes the lower bound above. The equality is neither assumed nor needed here.

5.2 Reflection and the joint moment

Let $r'_j(n)$ be the weighted count of $n = p + 2^{\nu_1} + \cdots + 2^{\nu_j}$ with $N^{1-\kappa_0} < p \le N$ and exponents in $[1, L]$. Let $r''_5(N)$ denote the left-hand side of (1.4). For fixed

$$a, b \ge 0, \quad a + b < 1, \quad 0 \le c < 1, \quad \tau > 0,$$

put

$$f_2 = r_2' - cLr_1', \qquad f_3 = r_3' - aLr_2' - bL^2 r_1', \qquad g_j = \max(f_j, 0),$$
$$F_2(t) = t^2 - cLt, \qquad F_3(t) = t^3 - aLt^2 - bL^2 t, \qquad Q(z) = z^2 - az - b,$$
$$\mathcal{I} = L^2 \int_{\mathbb{R}} /\mathbb{Z}|SF_2(G)|^2 + \tau^2 \int_{\mathbb{R}} /\mathbb{Z}|SF_3(G)|^2.$$

On $U_N = \{1 \le n < N : n \text{ odd}\}$, the prime number theorem and $3 \cdot 2^L = o(N)$ give

$$\sum_{U_N} g_2 \ge (1 - c + o(1))NL^2, \qquad \sum_{U_N} g_3 \ge (1 - a - b + o(1))NL^3.$$

Since $0 \le g_j \le r_j'$ and reflection permutes U_N, Cauchy's inequality and Parseval give

$$2\tau L r_5''(N) \ge \sum_{n \in U_N} (Lg_2(n) + \tau g_3(N-n))^2 - L^2 \sum_{U_N} g_2^2 - \tau^2 \sum_{U_N} g_3^2$$
$$\ge \frac{2}{N} \left(L \sum_{U_N} g_2 + \tau \sum_{U_N} g_3 \right)^2 - \mathcal{I}.$$

Thus

$$r_5''(N) \ge \frac{2NL^6(1 - c + \tau(1 - a - b))^2 - \mathcal{I}}{2\tau L} + o(NL^5). \tag{5.7}$$

Write $\boldsymbol{u} = (-c, 1, 0)^T$, $\boldsymbol{v} = (-b, -a, 1)^T$, and define

$$D = c^2 + \tau^2 b^2, \qquad H = \max_{z \in \mathcal{D}} (|z - c|^2 + \tau^2 |Q(z)|^2), \qquad K = \boldsymbol{u}^T A \boldsymbol{u} + \tau^2 \boldsymbol{v}^T A \boldsymbol{v}.$$

Since $0 \in \mathcal{D}$, $D \le H$. The mixed moments (5.3), with the zero shifts kept separate, yield

$$\mathcal{I}_{\mathfrak{M}} = 2NL^6((1 - c)^2 + \tau^2(1 - a - b)^2 + K + O(\eta) + o(1)) + DL^5 J.$$

The omitted zero-shift terms are $O(L^4 J) = o(NL^6)$ because $J \ll N \log N$. On $\mathfrak{m} \setminus \mathcal{E}$ the integrand is bounded by $HL^4|SG|^2$. On $\mathfrak{m} \cap \mathcal{E}$, the bound $|G| \le L$ and (5.1) give $o(NL^6)$. Combine these facts with (5.5) and (5.6). The sign $D - H \le 0$ is essential when using the lower bound for J. We obtain

$$\frac{\mathcal{I}}{2NL^6} \le (1 - c)^2 + \tau^2(1 - a - b)^2 + \mathcal{C} + O(\eta) + o(1), \tag{5.8}$$

where

$$\mathcal{C} = K + dD + BH, \qquad B = \frac{6.7814 - 2}{2} 1.279 + \frac{7 \log 2}{25}. \tag{5.9}$$

Here $1 + A_{11} < 1.279$ and $\log 2/2 - d = 7 \log 2/25$. Equations (5.7) and (5.8) prove that it suffices to have

$$\mathcal{C} < 2\tau(1 - c)(1 - a - b). \tag{5.10}$$

More precisely, the coefficient of NL^5 supplied by the argument is at least $\big(2\tau(1 - c)(1 - a - b) - \mathcal{C} - O(\eta) - o(1)\big)/\tau$.

6 Exact certification of the numerical inequality

Choose

$$a = \frac{1}{4}, \qquad b = \frac{7}{100}, \qquad c = \frac{23}{200}, \qquad \tau = 2. \tag{6.1}$$

Then

$$D = \frac{1313}{40000}, \qquad 2\tau(1 - c)(1 - a - b) = \frac{3009}{1250} = 2.4072.$$

6.1 The Gram contribution

Let $A_* = A(3)+A(5)+A(7)+A(15)+A(31)$. The exact finite calculations give $(A_*)_{11} > 0.2146$, $(A_*)_{22} > 0.0496$, and $(A_*)_{33} > 0.012$. By (5.4), the positive semidefinite tail $T = A - A_*$ satisfies $T_{11} < 0.254^2$, $T_{22} < 0.077^2$, $T_{33} < 0.04^2$. Hence for every real vector x,

$$x^T A x \le x^T A_* x + (0.254|x_1| + 0.077|x_2| + 0.04|x_3|)^2.$$

For the vectors from (6.1), the finite contributions are

q	$\boldsymbol{u}^T A(q)\boldsymbol{u} + 4\boldsymbol{v}^T A(q)\boldsymbol{v}$
3	30013/320000
5	5813/2560000
7	2237353/87480000
15	25763/1280000
31	282829/90000000

These entries can be reproduced using integer arithmetic: let $H_q = \{2^t \bmod q : 1 \le t \le \ell_q\}$, $n_0(s) = \mathbf{1}_{s=0}$, and $n_{j+1}(s) = \sum_{h\in H_q} n_j(s-h)$ modulo q. With $c_q(t) = \sum_{d|(q,t)} d\mu(q/d)$,

$$A_{ij}(q) = \frac{\sum_{s,t \bmod q} n_i(s) n_j(t) c_q(s-t)}{\varphi(q)^2 \ell_q^{i+j}}.$$

Their sum is 101411582429/699840000000; the combined tail bound is 350150477/10000000000. Therefore

$$K \le \frac{15739564176421}{87480000000000} < \frac{9}{50} = 0.18. \tag{6.2}$$

6.2 The polynomial maximum

The function $|z-c|^2 + 4|Q(z)|^2$ is subharmonic, so its maximum on the compact region $\mathcal{D}$ occurs on its boundary. On that boundary,

$$x \in [\alpha_-, \beta], \qquad u = |z|^2 = \frac{r}{1+r-x}.$$

The endpoints satisfy $[\alpha_-, \beta] \subset [-421/1000, 363/500]$; this follows directly by evaluating $x^2 - rx - r$ at the rational endpoints. Expanding the squared moduli gives the boundary value

$$h(x) = 4u^2 + (1 + 4a^2 + 8b - 8ax)u + c^2 + 4b^2 + (-2c + 8ab)x - 16bx^2.$$

After multiplication by the positive square $(1+r-x)^2$, the inequality $h(x) < 339/500$ is equivalent to $P(x) > 0$, where

$$P(x) = \frac{935932703}{167772160000} - \frac{1906686391}{10485760000}x + \frac{1119219663}{655360000}x^2 - \frac{3627}{1280}x^3 + \frac{28}{25}x^4. \tag{6.3}$$

Here $P(x) = (339/500 - h(x))(1+r-x)^2$ exactly. The following table is a rational positivity certificate. On each interval $[l, u]$, every degree-four Bernstein coefficient of $P(l + (u-l)t)$ exceeds the displayed positive bound.

l	u	Lower bound for every coefficient
$-421/1000$	$-537/4000$	$679/10000$
$-537/4000$	$73/8000$	$81/20000$
$73/8000$	$1439/32000$	$3/5000$
$1439/32000$	$1293/16000$	$81/1000000$
$1293/16000$	$61/400$	$589/1000000$
$61/400$	$363/500$	$277/500000$

For a direct check, if $P(x) = \sum_{k=0}^{4} p_k x^k$, first set

$$q_j = \sum_{k=j}^{4} p_k \binom{k}{j} l^{k-j}(u-l)^j, \qquad b_i = \sum_{j=0}^{i} q_j \frac{\binom{i}{j}}{\binom{4}{j}}.$$

Then $P(l + (u-l)t) = \sum_{i=0}^{4} b_i \binom{4}{i} t^i (1-t)^{4-i}$. The basis functions are nonnegative and sum to one on $[0,1]$. The table therefore proves, without a numerical maximization,

$$H < \frac{339}{500} = 0.678. \tag{6.4}$$

6.3 The margin and the representation count

Using (6.2), (6.4), and $\log 2 < 0.694$ in (5.9),

$$\begin{aligned} \mathcal{C} &< 0.18 + \frac{11 \cdot 0.694}{50}\frac{1313}{40000} + \left(\frac{4.7814 \cdot 1.279}{2} + \frac{7}{25}\,0.694\right) 0.678 \\ &= 2.3898848744 < 2.4072. \end{aligned}$$

The strict margin is 0.0173151256. After division by $\tau = 2$, the available coefficient is $0.0086575628 > 1/200$. Choose the fixed major-arc error η sufficiently small, then κ_0 as required by the original major-arc input, and finally N sufficiently large. The remaining $O(\eta)$ and $o(1)$ errors then leave a coefficient at least $1/200$ in (1.4). This completes the proof of Theorem 1.1.

7 Concluding remarks: the conditional five-power result

Five powers of 2 are enough under the hypotheses of Theorem 1.1. Every sufficiently large even integer then has a representation by two primes and five powers of 2, and the weighted number of ordered representations is at least $NL^5/200$. Taking the cited analytic results as stated, this conclusion holds under the family-wide zero bound $\Re\rho \le \Theta$ for any fixed $1/2 \le \Theta < 11/12$.

The improvement over the supplied paper is the weaker hypothesis sufficient for five powers. The proof combines the exceptional-set transfer and removal of smoothing with a strict numerical margin of 0.0173151256 in the reflection criterion. It yields no four-power conclusion. An unconditional five-power theorem would additionally require the external analytic inputs and an appropriate family-wide zero bound to be established; their proofs are outside the scope of this note.

The new deduction has been checked at the level of the explicit assumptions above: the annulus is handled by cancellation of identical nonzero-shift main terms; its zero shift is bounded separately; the endpoint sieve constant in (4.3) is independent of the smoothing width; and the smoothing limits are taken in the correct order. The finite Gram sums and all polynomial certificate inequalities were recomputed with exact rational arithmetic.

References

[1] H. Li and Z. Liu, *6 powers of 2 in the Linnik–Goldbach problem*, arXiv:2610.15784v1 (2026). Supplied source file `Arxiv.tex`. `https://arxiv.org/abs/2610.15784v1`.

[2] H. Li and Z. Liu, *Fourth moments of prime exponential sums and exceptional sets for sums and differences of primes*, arXiv:2610.26756v1 (2026), Theorem 1 and Proposition 4.1. `https://arxiv.org/abs/2610.26756v1`.

[3] B. Green, *Additive Combinatorics*, lecture notes, Section 2.3, "Selberg's sieve." `https://people.maths.ox.ac.uk/greenbj/papers/additive-combinatorics.pdf`.