A $\sqrt{n-1}$ Bound for Chains of Zeros of Successive Derivatives

Abstract

Let f be a complex polynomial of degree $n \geq 2$ whose zeros lie in the closed unit disk. We prove that one can select, for every $r = 0, \ldots, n-1$, a zero $z^{(r)}$ of $f^{(r)}$ so that the total length of the derivative chain

$$z^{(0)} \longrightarrow z^{(1)} \longrightarrow \cdots \longrightarrow z^{(n-1)}$$

is at most $\sqrt{n-1}$. More precisely, if u is the common centroid of the zeros of all derivatives, equivalently the unique zero of $f^{(n-1)}$, then the bound is

$$\sum_{r=0}^{n-2} \left| z^{(r)} - z^{(r+1)} \right| \leq \sqrt{(n-1)(1-|u|^2)}.$$

The proof combines Malamud's stochastic strengthening of the Gauss–Lucas theorem with an exact telescoping variance identity and Cauchy–Schwarz.

1 Statement

Let Z_r denote the multiset of zeros of $f^{(r)}$, counted with multiplicity. Thus

$$|Z_r| = n - r, \qquad 0 \leq r \leq n-1.$$

The last multiset Z_{n-1} consists of one point, which we denote by u.

Theorem 1 (Derivative-chain bound)**.** *Let $f \in \mathbb{C}[z]$ have degree $n \geq 2$, and suppose that every zero of f lies in the closed unit disk $\overline{\mathbb{D}}$. Then there are points*

$$z^{(r)} \in Z_r, \qquad r = 0, 1, \ldots, n-1,$$

such that

$$\sum_{r=0}^{n-2} \left| z^{(r)} - z^{(r+1)} \right| \leq \sqrt{(n-1)(1-|u|^2)}. \tag{1}$$

In particular,

$$\sum_{r=0}^{n-2} \left| z^{(r)} - z^{(r+1)} \right| \leq \sqrt{n-1}. \tag{2}$$

The point u is the centroid of the original zeros:

$$u = \frac{1}{n} \sum_{z \in Z_0} z, \tag{3}$$

where multiplicities are understood. This follows immediately from Vieta's formula (and is also the unique zero of $f^{(n-1)}$).

2 The one-step stochastic Gauss–Lucas lemma

The only non-elementary ingredient is a theorem of Malamud. We isolate exactly the consequence that is needed.

Lemma 1 (Stochastic Gauss–Lucas coupling)**.** *Let P be a complex polynomial of degree $m \geq 2$. Label its roots, with multiplicity, as*

$$\lambda_1, \ldots, \lambda_m,$$

and label the roots of P' as

$$\mu_1, \ldots, \mu_{m-1}.$$

Then there are numbers $a_{ji} \geq 0$ $(1 \leq j \leq m-1,\ 1 \leq i \leq m)$ such that

$$\sum_{i=1}^{m} a_{ji} = 1 \qquad (1 \leq j \leq m-1), \tag{4}$$

$$\sum_{j=1}^{m-1} a_{ji} = \frac{m-1}{m} \qquad (1 \leq i \leq m), \tag{5}$$

$$\mu_j = \sum_{i=1}^{m} a_{ji}\lambda_i \qquad (1 \leq j \leq m-1). \tag{6}$$

Consequently, there is a coupling of a uniformly chosen labeled root A of P and a uniformly chosen labeled root B of P' such that

$$\mathbb{E}(A \mid B) = B. \tag{7}$$

Proof. Put

$$\mu_m := \frac{1}{m}\sum_{i=1}^{m} \lambda_i.$$

The $k = 1$ case of Theorem 4.3 of Malamud [1] states that there exists an $m \times m$ unitary-stochastic (hence doubly stochastic) matrix

$$S = (s_{ji})_{1 \leq j,i \leq m}$$

such that

$$(\mu_1, \ldots, \mu_m)^T = S(\lambda_1, \ldots, \lambda_m)^T$$

and whose last row is constant:

$$s_{mi} = \frac{1}{m} \qquad (1 \leq i \leq m).$$

Set $a_{ji} = s_{ji}$ for $1 \leq j \leq m-1$. Since S is doubly stochastic, each of its rows and columns sums to 1. Deleting the last row therefore gives (4), (5), and (6).

Now let B be uniform on the labeled derivative roots $\mu_1, \ldots, \mu_{m-1}$. Conditional on the event that the label of B is j, choose $A = \lambda_i$ with probability a_{ji}. Equation (4) makes this a probability distribution, and (6) gives

$$\mathbb{E}(A \mid B = \mu_j) = \mu_j.$$

If derivative roots coincide, labels are retained; since all labels carrying the same value have the same conditional mean, (7) also holds for the complex-valued random variable B itself.

Finally, by (5),

$$\Pr(A = \lambda_i) = \frac{1}{m-1}\sum_{j=1}^{m-1} a_{ji} = \frac{1}{m}.$$

Thus A is uniform on the labeled roots of P, as claimed. □

3 Construction of the full chain

For every $r = 0, \dots, n-2$, apply Lemma 1 to the polynomial $f^{(r)}$, whose degree is $n-r$. This gives a transition kernel from a uniformly distributed labeled zero of $f^{(r+1)}$ to a uniformly distributed labeled zero of $f^{(r)}$.

Start with the deterministic point

$$Z_{n-1} = u,$$

and sample backwards, successively, according to these kernels. We obtain random variables

$$Z_0, Z_1, \dots, Z_{n-1}$$

with the following two properties:

1. Z_r is uniformly distributed on the labeled zeros of $f^{(r)}$;
2. for every $0 \le r \le n-2$,
$$\mathbb{E}(Z_r \mid Z_{r+1}) = Z_{r+1}. \tag{8}$$

Thus the sequence read in the reverse direction,

$$Z_{n-1}, Z_{n-2}, \dots, Z_0,$$

is a finite complex-valued martingale (with respect to the natural backwards filtration). For the proof below, the adjacent conditional expectation identities (8) are already sufficient.

4 Exact telescoping of the squared edge lengths

Fix $r \le n-2$. Since $Z_{r+1} - u$ is a function of Z_{r+1}, (8) gives

$$\begin{aligned}\mathbb{E}&\left[(Z_r - Z_{r+1})\overline{(Z_{r+1} - u)}\right] \\ &= \mathbb{E}\left[\mathbb{E}(Z_r - Z_{r+1} \mid Z_{r+1})\overline{(Z_{r+1} - u)}\right] = 0.\end{aligned}$$

Therefore

$$\mathbb{E}|Z_r - u|^2 = \mathbb{E}|Z_r - Z_{r+1}|^2 + \mathbb{E}|Z_{r+1} - u|^2. \tag{9}$$

Summing (9) from $r = 0$ to $n-2$ telescopes, and since $Z_{n-1} = u$, yields the exact identity

$$\boxed{\sum_{r=0}^{n-2} \mathbb{E}|Z_r - Z_{r+1}|^2 = \mathbb{E}|Z_0 - u|^2.} \tag{10}$$

Because Z_0 is uniform on the original zeros $z_1, \dots, z_n$ and u is their centroid,

$$\begin{aligned}\mathbb{E}|Z_0 - u|^2 &= \frac{1}{n}\sum_{i=1}^{n} |z_i - u|^2 \\ &= \frac{1}{n}\sum_{i=1}^{n} |z_i|^2 - |u|^2 \\ &\le 1 - |u|^2,\end{aligned} \tag{11}$$

since $|z_i| \le 1$ for every i. Combining (10) and (11),

$$\sum_{r=0}^{n-2} \mathbb{E}|Z_r - Z_{r+1}|^2 \le 1 - |u|^2. \tag{12}$$

5 Proof of Theorem 1

For each realization of the random chain, define

$$\ell := \sum_{r=0}^{n-2} |Z_r - Z_{r+1}|.$$

There are $n-1$ nonnegative summands. Hence Cauchy–Schwarz gives, pointwise,

$$\ell^2 \le (n-1) \sum_{r=0}^{n-2} |Z_r - Z_{r+1}|^2.$$

Taking expectations and using (12),

$$\mathbb{E}\ell^2 \le (n-1)(1-|u|^2). \tag{13}$$

The joint distribution has finite support, and each point in that support is an actual chain through zeros of the successive derivatives. At least one realization satisfies

$$\ell^2 \le \mathbb{E}\ell^2.$$

Together with (13), this gives

$$\ell \le \sqrt{(n-1)(1-|u|^2)},$$

which proves (1). Since $|u| \le 1$, (2) follows immediately. □

6 Remarks

Remark 1 (All degrees). The argument is uniform in n. In particular, it gives

$$L_n(f) \le \sqrt{n-1},$$

where $L_n(f)$ denotes the minimum total length of a complete chain through one zero of each of

$$f, f', \dots, f^{(n-1)}.$$

No form of Sendov's conjecture is used.

Remark 2 (Scaling). If all zeros lie in a disk of radius R centered at the origin, the same proof gives

$$L_n(f) \le \sqrt{(n-1)(R^2-|u|^2)} \le R\sqrt{n-1}.$$

More generally, after translating a disk of radius R centered at c to the origin, one obtains

$$L_n(f) \le \sqrt{(n-1)(R^2-|u-c|^2)}.$$

Remark 3 (Where the loss occurs). The stochastic Gauss–Lucas coupling and the variance identity (10) are exact. The only general loss in the final argument is the pointwise Cauchy–Schwarz estimate that turns the sum of squared edge lengths into ordinary path length. Any substantial improvement on the order $\sqrt{n}$ must therefore exploit additional geometric information about which edge-length patterns can actually occur along derivative chains.

References

[1] S. M. Malamud, *Inverse spectral problem for normal matrices and the Gauss–Lucas theorem*, Trans. Amer. Math. Soc. **357** (2005), no. 10, 4043–4064. DOI: 10.1090/S0002-9947-04-03649-9. See especially Theorem 4.3, $k = 1$.