# A Checked Improvement of the Numerical Constant in Section 5

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## Purpose

This note checks a small modification of the explicit numerical argument in Section 5 of the manuscript *The sum-product conjecture is false for real numbers.* The original Section 5 states that the construction gives

$$\max(|A+A|, |AA|) \leq |A|^{2-0.00000087}.$$

The modification below keeps the same construction

$$A = GP, \qquad G = B^{\times}(Y), \qquad P = X + B^{+}(\varepsilon X),$$

but improves the treatment of the product set and uses a sharper regulator estimate. The checked outcome is the following safe statement.

**Theorem 1.** *Using the same tower of totally real number fields and the same construction as in Section 5, but with the two changes described below, there are arbitrarily large finite sets $A \subset \mathbb{R}$ such that*

$$\max(|A+A|, |AA|) \leq |A|^{2-0.0007}.$$

*The optimized value suggested by the same calculation is about* 0.000719*, but* 0.0007 *is the safer quoted constant.*

## 1 The improved product-set estimate

Let $K$ be a totally real field of degree $d$, and let

$$H = \{x \in \mathbb{R}^d : x_1 + \cdots + x_d = 0\}.$$

Modulo the signs $\{\pm 1\}$, the logarithmic image of the unit group is a lattice $\Lambda \subset H$ of rank $d-1$. Put

$$K_Y = H \cap [-Y, Y]^d, \qquad S_Y = \Lambda \cap K_Y.$$

Then $B^{\times}(Y)$ maps exactly two-to-one onto $S_Y$.

**Lemma 1** (lattice doubling)**.** *For every $Y > 0$,*

$$|S_{2Y}| \leq 5^{d-1}|S_Y|.$$

*Consequently, if $G = B^{\times}(Y)$, then*

$$|GG| \leq 5^{d-1}|G|.$$

*Proof.* Choose a maximal set $T \subset S_{2Y}$ such that no two distinct elements of $T$ differ by an element of $S_Y$. Equivalently, for distinct $t, t' \in T$ we have $t - t' \notin K_Y \cap \Lambda$.

The translates

$$t + \frac{1}{2}K_Y, \qquad t \in T,$$

have disjoint interiors inside the affine space $H$. Indeed, if two such translates had intersecting interiors, then $t - t'$ would lie in the interior of $K_Y$, hence in $K_Y \cap \Lambda = S_Y$, a contradiction. Since $T \subset K_{2Y}$, all these translates lie in

$$K_{2Y} + \frac{1}{2}K_Y = \frac{5}{2}K_Y.$$

Comparing $(d-1)$-dimensional volumes gives

$$|T| \leq \frac{\mathrm{vol}_{d-1}((5/2)K_Y)}{\mathrm{vol}_{d-1}((1/2)K_Y)} = 5^{d-1}.$$

By maximality of $T$, every element of $S_{2Y}$ differs from some element of $T$ by an element of $S_Y$. Therefore

$$S_{2Y} \subset T + S_Y,$$

and hence $|S_{2Y}| \leq |T||S_Y| \leq 5^{d-1}|S_Y|$.

Finally, $GG$ maps at most two-to-one to $S_Y + S_Y \subset S_{2Y}$, whereas $G$ maps exactly two-to-one to $S_Y$. Hence

$$|GG| \leq 2|S_{2Y}| \leq 2 \cdot 5^{d-1}|S_Y| = 5^{d-1}|G|.$$

□

This replaces the cruder bound

$$|GG| \leq |B^\times(2Y)|$$

followed by a separation estimate. It is the main source of the improved constant.

## 2 A sharper regulator input

The proof quoted in the manuscript gives, for every $s > 1$,

$$2^d R_K h_K \leq 2s(s-1)\left(\pi^{-d/2}\Delta_K^{1/2}\right)^s \Gamma(s/2)^d \zeta(s)^d.$$

Assume now that the fields in the tower satisfy

$$\Delta_K \leq C_2^d, \qquad C_2 = 857.57.$$

Since $h_K \geq 1$, we obtain, for every fixed $s > 1$,

$$R_K \leq (1 + o(1))^d \left(\frac{\zeta(s)\Gamma(s/2)}{2\pi^{s/2}} C_2^{s/2}\right)^d.$$

The expression in parentheses is minimized near

$$s = 1.371966384\ldots,$$

where it is

$$101.9560759\ldots.$$

Thus, after passing to sufficiently large degrees in the tower, we may safely use

$$R_K \leq 103^d.$$

## 3 The construction and its estimates

Let

$$\phi = \frac{1+\sqrt{5}}{2}, \qquad \varepsilon = \frac{\phi - 1}{\phi + 1} = 0.2360679774\ldots.$$

This is the largest value allowed by the directness condition

$$\frac{1+\varepsilon}{1-\varepsilon} \le \phi.$$

Let

$$G = B^{\times}(Y), \qquad P = X + B^{+}(\varepsilon X), \qquad A = GP.$$

As in Section 5, the representation $a = up$ with $u \in G$ and $p \in P$ is unique. Indeed, if $u_1 p_1 = u_2 p_2$, then $u_1/u_2 = p_2/p_1$ is a unit, and every real embedding of $p_2/p_1$ lies in $(\phi^{-1}, \phi)$. The unit separation lemma therefore gives $u_1/u_2 \in \{\pm 1\}$, and the sign $-1$ is impossible because all embeddings of $p_2/p_1$ are positive. Hence

$$|A| = |G||P|.$$

Using $R_K \le 103^d$ and $\Delta_K \le C_2^d$, the lower bounds for the two lattices give, up to harmless $(1+o(1))^d$ factors,

$$|G| \ge \left(\frac{Y}{103}\right)^d, \qquad |P| \ge \left(\frac{\varepsilon X}{\sqrt{C_2}}\right)^d.$$

Therefore

$$|A| \ge (BXY)^d, \qquad B = \frac{\varepsilon}{103\sqrt{C_2}} = 0.0000782645\ldots.$$

For the product set, the improved lattice-doubling lemma gives

$$\begin{aligned} |AA| &\le |GG||PP| \\ &\le 5^{d-1}|G||P|^2 \\ &\le \left(\frac{515}{Y}\right)^d |A|^2, \end{aligned}$$

again with only harmless $(1+o(1))^d$ losses. Notice that here we keep the full unit ball $G$; we do not need to discard elements of $G$.

For the sumset, every element of $A$ has every real embedding bounded in absolute value by $(1+\varepsilon)Xe^Y$. Hence

$$A + A \subset B^{+}(2(1+\varepsilon)Xe^Y),$$

and the additive lattice upper bound gives

$$|A + A| \le (4(1+\varepsilon)Xe^Y + 1)^d.$$

Dividing by the lower bound for $|A|^2$ yields

$$|A + A| \le \left(\frac{M_1 e^Y}{XY^2} + \frac{M_2}{X^2 Y^2}\right)^d |A|^2,$$

where

$$M_2 = \frac{103^2 C_2}{\varepsilon^2} = 163256270.4\ldots,$$

and

$$M_1 = 4(1+\varepsilon)M_2 = 807183391.9\ldots.$$

We shall also need an upper bound for $|A|$ in order to translate a factor of the form $q^d$ into a power saving in $|A|$. The standard upper bounds give

$$|G| \le 10(5Y+1)^{d-1}, \qquad |P| \le (2\varepsilon X+1)^d.$$

Thus, for fixed $X, Y$ and sufficiently large $d$,

$$|A| \le \exp(Ld),$$

where $L$ may be taken to be any number larger than

$$\log(5Y+1) + \log(2\varepsilon X+1).$$

# 4 Numerical choice

Take

$$Y = 1415, \qquad X = \lfloor e^{1423} \rfloor.$$

Then the product estimate gives

$$\frac{515}{Y} = 0.3639575971\ldots < 0.364.$$

The first additive factor satisfies

$$\frac{M_1 e^Y}{XY^2} < 0.136,$$

and the second additive factor is negligible. Thus both estimates are bounded by

$$(0.364 + o(1))^d |A|^2.$$

On the other hand, the upper bound for $|A|$ gives, for all sufficiently large $d$,

$$|A| \le e^{1432d}.$$

Finally,

$$-\log(0.364) = 1.010601\ldots \quad \text{and} \quad 0.0007 \cdot 1432 = 1.0024.$$

Therefore, after absorbing the harmless $(1+o(1))^d$ factors by taking $d$ sufficiently large,

$$(0.364 + o(1))^d \le |A|^{-0.0007}.$$

Consequently

$$\max(|A+A|, |AA|) \le |A|^{2-0.0007}.$$

# Conclusion

The recheck is positive. The improvement rests on two valid changes:

1. use the lattice-doubling estimate $|GG| \le 5^{d-1}|G|$ instead of bounding $GG$ directly by the full ball $B^\times(2Y)$; and

2. optimize the regulator estimate over $s > 1$, which allows the safe bound $R_K \le 103^d$ along the same tower for all sufficiently large degrees.

Together these changes improve the explicit constant from 0.00000087 to the safe value 0.0007.