A Note on the Erdős–Szemerédi LCM-Window Problem for Exponents Greater Than 1/4

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Abstract

Let $A = \{a_1 < a_2 < \cdots\} \subseteq \mathbb{N}$, and let $F(A, X, k)$ denote the number of indices i for which

$$\operatorname{lcm}(a_i, a_{i+1}, \ldots, a_{i+k-1}) < X.$$

Using a recent divisor-in-short-interval estimate of Letendre, we prove that for every $\varepsilon > 1/4$ there is a fixed $k = k(\varepsilon)$ such that

$$F(A, X, k) \ll_{\varepsilon,k} X^{\varepsilon}$$

uniformly in A.

1 Input from divisors in short intervals

For an integer $n \geq 1$, write

$$D_n(U, H) = \#\{d \mid n : U \leq d \leq U + H\}.$$

We use the following proposition of Letendre [1, Proposition 1].

Proposition 1 (Letendre)**.** *Let $n \geq 1$ be fixed, and let $0 < \theta < 1$ and $0 < \eta < \theta^2$. Then*

$$D_n(n^{\theta}, n^{\theta^2-\eta}) \ll \frac{\theta(1-\theta)}{\eta} + \frac{1}{\theta(1-\theta)}.$$

In particular, for every $0 < \lambda < 1/2$ and every $0 < \eta < \lambda^2$, there is a constant $C(\lambda, \eta)$ such that

$$D_n(n^{\theta}, n^{\theta^2-\eta}) \leq C(\lambda, \eta)$$

for all $n \geq 1$ and all $\lambda \leq \theta \leq 1 - \lambda$.

2 Two elementary packing lemmas

For a good index i, put

$$L_i = \operatorname{lcm}(a_i, a_{i+1}, \ldots, a_{i+k-1}), \qquad u_i = a_i, \qquad H_i = a_{i+k-1} - a_i.$$

Then $L_i < X$, and the interval $[u_i, u_i + H_i]$ contains the k divisors

$$a_i, a_{i+1}, \ldots, a_{i+k-1}$$

of L_i.

Lemma 1 (Disjointness after passing to one residue class)**.** *Fix $k \geq 1$ and one residue class modulo k. The intervals*

$$[a_i, a_{i+k-1}]$$

attached to good indices i in this residue class are pairwise disjoint.

Proof. If $i < j$ and $i \equiv j \pmod{k}$, then $j \geq i + k$. Since the sequence A is strictly increasing,

$$a_{i+k-1} < a_j.$$

Thus the interval attached to i lies strictly to the left of the interval attached to j. □

Lemma 2 (Packing by separated left endpoints)**.** *Let $\mathcal{I}$ be a finite family of pairwise disjoint intervals $[u, u + H]$ with integer left endpoints $u \in [U, 2U)$. If every interval in $\mathcal{I}$ has length at least $h > 0$, then*

$$\#\mathcal{I} \ll \frac{U}{h} + 1,$$

with an absolute implied constant.

Proof. Order the intervals by increasing left endpoint. If two consecutive intervals have left endpoints $u < v$, then disjointness and the lower bound on the first length give

$$v \geq u + H \geq u + h,$$

up to an immaterial endpoint convention. Hence the left endpoints are h-separated inside an interval of length U, and there are $O(U/h + 1)$ of them. □

3 Proof of the $\varepsilon > 1/4$ bound

Theorem 1. *For every $\varepsilon > 1/4$, there exists an integer $k = k(\varepsilon)$ such that, uniformly for every increasing sequence $A \subseteq \mathbb{N}$,*

$$F(A, X, k) \ll_{\varepsilon,k} X^{\varepsilon}.$$

Consequently, for every $\varepsilon > 1/4$, the Erdős–Szemerédi inequality

$$F(A, X, k) < X^{\varepsilon}$$

holds for all sufficiently large X, with k depending only on ε.

Proof. Fix $\varepsilon > 1/4$. Choose λ and η such that

$$0 < \lambda < \frac{1}{2}, \qquad \frac{\lambda}{1-\lambda} < \varepsilon, \qquad 0 < \eta < \min\{\lambda^2, \varepsilon - 1/4\}.$$

Let $C(\lambda, \eta)$ be the uniform constant from Proposition 1, and choose an integer

$$k > \max\{2, C(\lambda, \eta) + 1\}.$$

We shall prove that this k works.

By Lemma 1, it is enough to bound the number of good indices in a fixed residue class modulo k and then multiply by k. Hence, throughout the proof, restrict to one residue class modulo k; all associated intervals $[u_i, u_i + H_i]$ are pairwise disjoint.

For a good index i, define

$$L_i = \operatorname{lcm}(a_i, a_{i+1}, \ldots, a_{i+k-1}) < X.$$

Since $u_i = a_i$ divides L_i, and since $u_i \leq L_i$, we may write

$$u_i = L_i^{\theta_i}$$

for some $0 \leq \theta_i \leq 1$, except for harmless cases with $u_i = 1$ or $L_i = 1$. These exceptional cases contribute only $O_k(1)$ indices: $u_i = 1$ can occur for at most one index, and $L_i = 1$ is impossible for $k \geq 2$ in a strictly increasing sequence of positive integers.

We split the remaining good indices into three classes.

1. Small θ_i

Suppose $\theta_i < \lambda$. Then

$$u_i = L_i^{\theta_i} < X^{\lambda}.$$

There are at most $O(X^{\lambda})$ possible integer values of u_i, so this contribution is $O(X^{\lambda}) = O(X^{\varepsilon})$.

2. Large θ_i

Suppose $\theta_i > 1 - \lambda$. We use a simple spacing estimate for divisors near the top of L_i. If $d < e$ are two distinct divisors of L_i, write $d = L_i/r$ and $e = L_i/s$ with integers $r > s \geq 1$. Then

$$e - d = L_i \left(\frac{1}{s} - \frac{1}{r} \right) = \frac{L_i(r-s)}{rs} \geq \frac{L_i}{r^2} = \frac{d^2}{L_i}.$$

In our interval all listed divisors are at least $u_i = L_i^{\theta_i}$, so consecutive listed divisors in the interval are separated by at least

$$\frac{u_i^2}{L_i} = u_i^{2-1/\theta_i} \geq u_i^{(1-2\lambda)/(1-\lambda)},$$

because $\theta_i > 1 - \lambda$ and the function $2 - 1/t$ is increasing for $t > 0$.

Group intervals by dyadic ranges $U \leq u_i < 2U$. In such a range, every interval of the large-θ_i type has length

$$H_i \gg_k U^{(1-2\lambda)/(1-\lambda)},$$

because it contains k listed divisors and hence $k - 1$ gaps of the above size. By Lemma 2, the number of such intervals in this dyadic range is

$$O_k \left(\frac{U}{U^{(1-2\lambda)/(1-\lambda)}} + 1 \right) = O_k \left(U^{\lambda/(1-\lambda)} + 1 \right).$$

Summing over dyadic $1 \leq U < X$ gives

$$O_k \left(X^{\lambda/(1-\lambda)} \log X \right) = O_{\varepsilon,k}(X^{\varepsilon}),$$

because $\lambda/(1-\lambda) < \varepsilon$.

3. Middle range $\lambda \leq \theta_i \leq 1 - \lambda$

It remains to treat good indices with

$$\lambda \leq \theta_i \leq 1 - \lambda.$$

We use the uniform compact-range consequence of Proposition 1. Since $k > C(\lambda, \eta)$, the interval $[u_i, u_i + H_i]$ cannot have length less than $L_i^{\theta_i^2 - \eta}$. Indeed, if it had smaller length, then the larger divisor-counting interval

$$[L_i^{\theta_i}, L_i^{\theta_i} + L_i^{\theta_i^2 - \eta}]$$

would contain the same k listed divisors of L_i, contradicting

$$D_{L_i}(L_i^{\theta_i}, L_i^{\theta_i^2 - \eta}) \leq C(\lambda, \eta) < k.$$

Thus

$$H_i \geq L_i^{\theta_i^2 - \eta}.$$

Since $u_i = L_i^{\theta_i}$, this gives

$$H_i \geq u_i^{\theta_i} L_i^{-\eta} > u_i^{\theta_i} X^{-\eta}.$$

Again group indices dyadically, say

$$U \le u_i < 2U, \qquad \alpha = \frac{\log U}{\log X}, \qquad 0 \le \alpha \le 1.$$

Because $u_i \ge U$ and $L_i < X$, we have

$$\theta_i = \frac{\log u_i}{\log L_i} \ge \frac{\log U}{\log X} = \alpha.$$

Therefore

$$u_i^{\theta_i} \ge U^{\theta_i} \ge U^{\alpha} = X^{\alpha^2},$$

and hence

$$H_i \ge X^{\alpha^2 - \eta}.$$

By Lemma 2, the number of middle-range intervals with $U \le u_i < 2U$ is

$$O\left(\frac{U}{X^{\alpha^2-\eta}} + 1\right) = O\left(X^{\alpha-\alpha^2+\eta} + 1\right).$$

Since

$$\alpha - \alpha^2 \le \frac{1}{4}$$

for all real α, this is

$$O\left(X^{1/4+\eta} + 1\right).$$

Summing over the $O(\log X)$ dyadic choices of U yields

$$O\left(X^{1/4+\eta} \log X\right) = O_\varepsilon(X^\varepsilon),$$

because $1/4 + \eta < \varepsilon$.

Combining the three cases, the number of good indices in the chosen residue class is $O_{\varepsilon,k}(X^\varepsilon)$. Multiplying by the k residue classes modulo k gives

$$F(A, X, k) \ll_{\varepsilon,k} X^\varepsilon,$$

as claimed.

For the final displayed inequality with the strict exponent ε, apply the just-proved big-O statement with any exponent ε' satisfying $1/4 < \varepsilon' < \varepsilon$. Since the implied constant is uniform in A, we have $C_{\varepsilon',k} X^{\varepsilon'} < X^\varepsilon$ for all sufficiently large X. □

Remark 1. *The endpoint $\varepsilon = 1/4$ is not obtained by this argument. The exponent $1/4$ appears from the elementary maximum*

$$\max_{0 \le \alpha \le 1} (\alpha - \alpha^2) = \frac{1}{4}.$$

The example $A = \mathbb{N}$ also gives the lower obstruction $F(\mathbb{N}, X, k) \gg_k X^{1/k}$. Thus the proof above reaches every exponent strictly larger than $1/4$, but not the endpoint.

References

[1] Patrick Letendre, *Divisors of an Integer in a Short Interval*, arXiv:2503.12146, 2025.