Rigorous Proof of Mutual Singularity for the Φ^4_3 Measure

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Abstract

We prove that the Φ^4_3 measure μ on the three-dimensional torus $\mathbb{T}^3$ is mutually singular with respect to its pushforward $(T_\psi)_*\mu$ under any non-zero smooth shift ψ. The proof relies on the construction of a separating event using a renormalised cubic observable at super-exponential scales. We rigorously establish that while the observable vanishes for μ, the shift induces a deterministic drift in the linear counterterm that diverges in probability, driving the singularity.

1 Problem Statement

Let $\mathbb{T}^3$ be the three-dimensional unit torus and let μ be the Φ^4_3 measure on the space of distributions $\mathcal{D}'(\mathbb{T}^3)$. Let $\psi \in C^\infty(\mathbb{T}^3)$ be a non-zero smooth function. The shift map is defined as $T_\psi(\Phi) = \Phi + \psi$. We prove that μ and $(T_\psi)_*\mu$ are mutually singular.

2 Theorem

Theorem 1. *For every non-zero $\psi \in C^\infty(\mathbb{T}^3)$, the measures μ and $(T_\psi)_*\mu$ are mutually singular ($\mu \perp (T_\psi)_*\mu$).*

Proof. **1. Exact Definitions and Constants.** To invoke the results of Hairer [1] rigorously, we explicitly adopt the definitions used therein. Let $\rho \in C_c^\infty(\mathbb{R}^3)$ be a smooth, even mollifier with $\int \rho = 1$. Let a, b be the specific renormalisation constants associated with this mollifier as defined in [1, Eq. 1.1]. A crucial property of the Φ^4_3 theory is that the linear mass renormalization constant b is non-zero ($b \neq 0$).

We define a sequence of super-exponentially small scales:

$$\varepsilon_n = \exp(-e^n), \quad n \in \mathbb{N}. \tag{1}$$

Let $\Phi_n = \Phi * \rho_{\varepsilon_n}$ and $\psi_n = \psi * \rho_{\varepsilon_n}$, where $\rho_\varepsilon(x) = \varepsilon^{-3}\rho(x/\varepsilon)$.

We define the renormalised cubic observable $X_n(f)$ for a test function $f \in C^\infty(\mathbb{T}^3)$:

$$X_n(f) = e^{-3n/4} \left\langle \Phi_n^3 - C_1(\varepsilon_n)\Phi_n - C_2(\varepsilon_n)\Phi, f \right\rangle. \tag{2}$$

The coefficients C_1, C_2 are defined exactly as:

$$C_1(\varepsilon_n) = \frac{3a}{\varepsilon_n}, \qquad C_2(\varepsilon_n) = 9b\log(\varepsilon_n^{-1}). \tag{3}$$

With our choice of scale ε_n, we have the exact relation $C_2(\varepsilon_n) = 9be^n$.

2. The Separating Event. We define the event A_f as:

$$A_f = \left\{ \Phi \in \mathcal{D}'(\mathbb{T}^3) : \lim_{n\to\infty} X_n(f)(\Phi) = 0 \right\}. \tag{4}$$

The following lemma is a direct consequence of Hairer's main result.

Lemma 2 (Hairer [1], Thm 1.1). *For any $f \in C^\infty(\mathbb{T}^3)$, $\mu(A_f) = 1$.*

3. Analysis of the Shifted Measure. We claim that $(T_\psi)_*\mu(A_f) = 0$. This is equivalent to showing that for μ-almost every Φ, the shifted field $\Phi + \psi$ does not belong to A_f. Evaluating $X_n(f)$ on the shifted field $\Phi + \psi$:

$$X_n(f)(\Phi + \psi) = e^{-3n/4} \left\langle (\Phi_n + \psi_n)^3 - C_1(\varepsilon_n)(\Phi_n + \psi_n) - C_2(\varepsilon_n)(\Phi + \psi), f \right\rangle \tag{5}$$

$$= X_n(f)(\Phi) + R_n(\Phi, \psi). \tag{6}$$

We expand the cubic term and group the result to isolate the Wick-ordered square $:\Phi_n^2: = \Phi_n^2 - \frac{1}{3}C_1(\varepsilon_n) = \Phi_n^2 - a\varepsilon_n^{-1}$.

$$R_n(\Phi, \psi) = \underbrace{e^{-3n/4}\langle 3(\Phi_n^2 - a\varepsilon_n^{-1})\psi_n, f\rangle}_{Y_n^{(1)}} \quad \text{(Quadratic Fluctuation)} \tag{7}$$

$$+ \underbrace{e^{-3n/4}\langle 3\Phi_n\psi_n^2, f\rangle}_{Y_n^{(2)}} \quad \text{(Linear Fluctuation)} \tag{8}$$

$$+ \underbrace{e^{-3n/4}\langle \psi_n^3, f\rangle}_{Z_n} - \underbrace{e^{-3n/4}C_2(\varepsilon_n)\langle \psi, f\rangle}_{\text{Drift}}. \tag{9}$$

4. Control of Fluctuations.

Lemma 3 (Tightness of Fluctuations). *For any smooth f, ψ, the random terms $Y_n^{(1)}$ and $Y_n^{(2)}$ converge to 0 in probability as $n \to \infty$ with respect to μ.*

Proof. It is a standard result in the construction of the Φ^4_3 measure (see [1] and references therein regarding the regularity structures construction) that Φ and $:\Phi^2:$ exist as well-defined random distributions in the Hölder-Besov spaces $\mathcal{C}^{-1/2-\kappa}$ and $\mathcal{C}^{-1-\kappa}$ respectively, for any $\kappa > 0$. For any random distribution Z in $\mathcal{C}^\alpha$, the mollified sequence $Z * \rho_{\varepsilon_n}$ converges to Z in $\mathcal{C}^{\alpha-\delta}$ (and thus in distribution when paired with smooth functions). Consequently, the sequences of scalar random variables $U_n = \langle :\Phi_n^2:, \psi_n f\rangle$ and $V_n = \langle \Phi_n, \psi_n^2 f\rangle$ converge in distribution to $\langle :\Phi^2:, \psi f\rangle$ and $\langle \Phi, \psi^2 f\rangle$. Convergent sequences are tight (bounded in probability). Since $Y_n^{(1)} = e^{-3n/4}U_n$ and $Y_n^{(2)} = e^{-3n/4}V_n$, and the factor $e^{-3n/4}$ vanishes deterministically, the products converge to 0 in probability. □

5. The Deterministic Drift. The behavior of the shifted observable is dominated by the linear counterterm $C_2(\varepsilon_n)$. Substituting $C_2(\varepsilon_n) = 9be^n$:

$$\text{Drift}_n = -e^{-3n/4}(9be^n)\langle \psi, f\rangle = -9be^{n/4}\langle \psi, f\rangle. \tag{10}$$

Since ψ is non-zero and $b \neq 0$, we choose $f = \psi$ (which is smooth). Then $\langle \psi, f\rangle = \|\psi\|_{L^2}^2 > 0$. For this choice, the drift term diverges:

$$\lim_{n\to\infty} |\text{Drift}_n| = \infty. \tag{11}$$

6. Conclusion. Combining these results, we have:

$$X_n(f)(\Phi + \psi) = \underbrace{X_n(f)(\Phi)}_{\to 0 \text{ a.s.}} + \underbrace{Y_n^{(1)} + Y_n^{(2)}}_{\to 0 \text{ in prob.}} + \underbrace{Z_n}_{\to 0} - \underbrace{9be^{n/4}\langle \psi, f\rangle}_{\to -\infty}. \tag{12}$$

Consequently, $|X_n(f)(\Phi + \psi)| \to \infty$ in probability with respect to μ. This implies that the measure of the set where the limit is 0 must be 0:

$$(T_\psi)_*\mu(A_f) = \mu\left(\left\{\Phi : \lim_{n\to\infty} X_n(f)(\Phi + \psi) = 0\right\}\right) \tag{13}$$

$$\leq \mu\left(\liminf_{n\to\infty}\{|X_n(f)(\Phi + \psi)| \leq 1\}\right) \tag{14}$$

$$\leq \lim_{n\to\infty} \mu(|X_n(f)(\Phi + \psi)| \leq 1) = 0. \tag{15}$$

Since $\mu(A_f) = 1$ and $(T_\psi)_*\mu(A_f) = 0$, the measures are mutually singular. □

References

[1] M. Hairer. *Φ^4_3 is orthogonal to GFF*. Note dated September 16, 2022. Available at `https://hairer.org/Phi4.pdf`.