Proof for Question 2 (after applying Review 6)

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Abstract

Let F be a non-archimedean local field. For generic irreducible admissible representations Π of $\mathrm{GL}_{n+1}(F)$ and π of $\mathrm{GL}_n(F)$, we construct Whittaker functions $W \in \mathcal{W}(\Pi, \psi^{-1})$ and $V \in \mathcal{W}(\pi, \psi)$ such that the twisted local Rankin–Selberg integral

$$I(s, W, V) = \int_{N_n \backslash \mathrm{GL}_n(F)} W(\mathrm{diag}(g, 1)u_Q)\, V(g)\, |\det g|^{s-\frac{1}{2}}\, dg$$

is absolutely convergent and nonzero for all $s \in \mathbb{C}$.

1 Notation and setup

Let F be a non-archimedean local field with ring of integers $\mathfrak{o}$, maximal ideal $\mathfrak{p}$, and residue field of size q. For $r \geq 1$, let $N_r \subset \mathrm{GL}_r(F)$ be the subgroup of upper-triangular unipotent matrices.

Fix a *nontrivial* additive character $\psi : F \to \mathbb{C}^\times$, and view it as a character of N_r by

$$\psi(u) = \psi\left(\sum_{i=1}^{r-1} u_{i,i+1}\right).$$

Let $c = c(\psi) \geq 0$ be such that ψ is trivial on $\mathfrak{p}^c$ (and nontrivial on $\mathfrak{p}^{c-1}$ if $c > 0$); only triviality on $\mathfrak{p}^c$ will be used.

Let Π be a generic irreducible admissible representation of $\mathrm{GL}_{n+1}(F)$ realized in its ψ^{-1}-Whittaker model $\mathcal{W}(\Pi, \psi^{-1})$, and let π be a generic irreducible admissible representation of $\mathrm{GL}_n(F)$ realized in $\mathcal{W}(\pi, \psi)$.

Fix $Q \in F^\times$ and set

$$u_Q := I_{n+1} + QE_{n,n+1} \in \mathrm{GL}_{n+1}(F),$$

where $E_{n,n+1}$ is the standard matrix unit.

Measures. Fix Haar measures $d\tilde{g}$ on $\mathrm{GL}_n(F)$ and du on $N_n(F)$. Let dg denote the associated quotient measure on $N_n \backslash \mathrm{GL}_n(F)$, characterized by

$$\int_{\mathrm{GL}_n(F)} \Phi(\tilde{g})\, d\tilde{g} = \int_{N_n \backslash \mathrm{GL}_n(F)} \int_{N_n(F)} \Phi(u\tilde{g})\, du\, dg \qquad (\Phi \in C_c^\infty(\mathrm{GL}_n(F))). \tag{1}$$

For $W \in \mathcal{W}(\Pi, \psi^{-1})$ and $V \in \mathcal{W}(\pi, \psi)$ define the twisted Rankin–Selberg integral

$$I(s, W, V) = \int_{N_n \backslash \mathrm{GL}_n(F)} W(\mathrm{diag}(g, 1) u_Q)\, V(g)\, |\det g|^{s-\frac{1}{2}}\, dg. \tag{2}$$

2 Mirabolic restriction

Let $P_{n+1} \subset \mathrm{GL}_{n+1}(F)$ be the *mirabolic* subgroup: matrices whose last row is $(0, \ldots, 0, 1)$. Write $P := P_{n+1}$ and $N := N_{n+1} \cap P$.

Definition 1 (Induced model)**.** We realize $\mathrm{Ind}_N^P(\psi^{-1})$ as the space of locally constant functions $\Phi : P \to \mathbb{C}$ such that

$$\Phi(np) = \psi^{-1}(n)\Phi(p) \qquad (n \in N,\ p \in P),$$

and whose support is compact modulo N (equivalently, the image of $\mathrm{supp}(\Phi)$ in $N \backslash P$ is compact).

Lemma 1 (Mirabolic restriction / Kirillov model)**.** *If Π is generic, then the restriction map*

$$\mathrm{res}_P : \ \mathcal{W}(\Pi, \psi^{-1}) \longrightarrow \mathrm{Ind}_N^P(\psi^{-1})$$

has image containing $C_c^\infty(N \backslash P)$ (viewed as functions on the quotient). Equivalently, any compactly supported, locally constant function on $N \backslash P$ can be realized (as in Definition 1) by the restriction of some Whittaker function $W \in \mathcal{W}(\Pi, \psi^{-1})$.

Remark 1. This is the standard Kirillov model statement for GL_{n+1} (Bernstein–Zelevinsky; Jacquet–Piatetski-Shapiro–Shalika). We only use the stated consequence: the ability to prescribe compactly supported data on the mirabolic quotient.

3 Choice of V and a compact open subset of $N_n \backslash \mathrm{GL}_n$

Choose a nonzero Whittaker functional $\lambda : \pi \to \mathbb{C}$, so $\lambda(\pi(u)w) = \psi(u)\lambda(w)$ for all $u \in N_n$. Pick $v \in \pi$ with $\lambda(v) \neq 0$. Since π is smooth, there exists a compact open subgroup $K \subset \mathrm{GL}_n(F)$ such that v is K–fixed.

We now choose a *smaller* compact open subgroup K_n with three properties: (i) it fixes v, (ii) it lies in $\mathrm{GL}_n(\mathfrak{o})$ (so $|\det| \equiv 1$ on it), and (iii) ψ is trivial on $N_n \cap K_n$. Set

$$K_n := \Big(K \cap \mathrm{GL}_n(\mathfrak{o})\Big) \cap \Big(1 + \mathfrak{p}^{\,c} M_n(\mathfrak{o})\Big).$$

Then K_n is compact open, v is K_n–fixed, $K_n \subset \mathrm{GL}_n(\mathfrak{o})$, and any $u \in N_n \cap K_n$ has all superdiagonal entries in $\mathfrak{p}^{\,c}$, hence $\psi(u) = 1$ by the choice of $c(\psi)$.

Define

$$V(g) := \lambda(\pi(g)v).$$

Then for $u \in N_n$,

$$V(ug) = \lambda(\pi(u)\pi(g)v) = \psi(u)V(g),$$

so $V \in \mathcal{W}(\pi, \psi)$. Moreover V is right K_n–invariant, hence $V(k) = V(1) = \lambda(v) \neq 0$ for all $k \in K_n$.

Define the compact open subset

$$U := (N_n \cap K_n)\backslash K_n \ \subset \ N_n\backslash \mathrm{GL}_n(F).$$

Since $K_n \subset \mathrm{GL}_n(\mathfrak{o})$, we have $|\det k| = 1$ for $k \in K_n$, hence $|\det g| = 1$ for $g \in U$.

4 Identifying the mirabolic quotient with $N_n\backslash \mathrm{GL}_n$

Lemma 2 (Mirabolic quotient equals $N_n\backslash \mathrm{GL}_n$)**.** *Write elements of P in block form*

$$p = \begin{pmatrix} A & b \\ 0 & 1 \end{pmatrix}, \qquad A \in \mathrm{GL}_n(F),\ b \in F^n.$$

Then the map

$$\theta : \ N\backslash P \ \longrightarrow \ N_n\backslash \mathrm{GL}_n(F), \qquad Np \ \longmapsto \ N_nA$$

is a well-defined homeomorphism of locally compact totally disconnected spaces, with inverse

$$\theta^{-1}(N_nA) = N \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}.$$

Proof. Let $p = \begin{pmatrix} A & b \\ 0 & 1 \end{pmatrix} \in P$, $A \in \mathrm{GL}_n(F)$, $b \in F^n$.

Well-defined. Any element of N has the form $\begin{pmatrix} u & x \\ 0 & 1 \end{pmatrix}$ with $u \in N_n$ and $x \in F^n$. Then

$$\begin{pmatrix} u & x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} A & b \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} uA & ub + x \\ 0 & 1 \end{pmatrix},$$

so the upper-left block changes from A to uA. Hence the class $N_nA \in N_n\backslash \mathrm{GL}_n(F)$ depends only on the coset $Np \in N\backslash P$, proving θ is well-defined.

Surjectivity. Given $A \in \mathrm{GL}_n(F)$, the element $\begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix} \in P$ maps to N_nA.

Injectivity. Suppose $\theta(Np) = \theta(Np')$ with $p = \begin{pmatrix} A & b \\ 0 & 1 \end{pmatrix}$ and $p' = \begin{pmatrix} A' & b' \\ 0 & 1 \end{pmatrix}$. Then $N_nA = N_nA'$, so $A' = uA$ for some $u \in N_n$. Let $x := b' - ub$. Then $\begin{pmatrix} u & x \\ 0 & 1 \end{pmatrix} \in N$ and $\begin{pmatrix} u & x \\ 0 & 1 \end{pmatrix} p = p'$, so $Np = Np'$.

Topology. Define $\theta^{-1}(N_n A) = N\begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}$. This is well-defined because replacing A by uA with $u \in N_n$ corresponds to left-multiplying by $\mathrm{diag}(u,1) \in N$. Both θ and θ^{-1} are induced by continuous block projection/embedding maps, hence are continuous. Since they are inverse bijections, θ is a homeomorphism. □

5 Construction of W

For $g \in \mathrm{GL}_n(F)$, note that $\mathrm{diag}(g,1) \in P$ and $u_Q \in P$ (both have last row $(0,\ldots,0,1)$), hence

$$\mathrm{diag}(g,1)u_Q \in P \qquad \text{for all } g \in \mathrm{GL}_n(F). \tag{3}$$

Define the compact subset of P

$$\Omega := \{\mathrm{diag}(k,1)u_Q : \ k \in K_n\} \subset P,$$

and let $\overline{\Omega} \subset N\backslash P$ be its image.

Lemma 3. *Under θ from Lemma 2, the subset $\overline{\Omega}$ corresponds to $U = (N_n \cap K_n)\backslash K_n \subset N_n\backslash\,\mathrm{GL}_n(F)$. In particular, $\overline{\Omega}$ is compact open.*

Proof. For $k \in K_n$, the element $\mathrm{diag}(k,1)u_Q$ has upper-left block k, so $\theta(N\,\mathrm{diag}(k,1)u_Q) = N_n k$. Thus $\theta(\overline{\Omega}) = U$. Since U is compact open and θ is a homeomorphism, $\overline{\Omega}$ is compact open. □

5.1 Cutoff function on $N\backslash P$

Define

$$f := \lambda(v)^{-1} \cdot \mathbf{1}_{\overline{\Omega}} \ \in \ C_c^\infty(N\backslash P).$$

5.2 An explicit induced vector $\widetilde{f} \in \mathrm{Ind}_N^P(\psi^{-1})$

We define $\widetilde{f} : P \to \mathbb{C}$ as an induced-model vector supported on $N\Omega$.

Definition 2. Define $\widetilde{f} : P \to \mathbb{C}$ by

$$\widetilde{f}(p) = \begin{cases} \psi^{-1}(n)\,\lambda(v)^{-1}, & \text{if } p = n\omega \text{ for some } n \in N,\ \omega \in \Omega, \\ 0, & \text{if } p \notin N\Omega. \end{cases}$$

Lemma 4 (Well-definedness and membership in the induced model)**.** *The function $\widetilde{f}$ of Definition 2 is well-defined, lies in $\mathrm{Ind}_N^P(\psi^{-1})$ in the sense of Definition 1, and satisfies:*

1. $\mathrm{supp}(\widetilde{f}) \subset N\Omega$ *and the image of* $\mathrm{supp}(\widetilde{f})$ *in* $N\backslash P$ *equals* $\overline{\Omega}$*;*
2. *for all $k \in K_n$, we have $\widetilde{f}(\mathrm{diag}(k,1)u_Q) = \lambda(v)^{-1}$.*

Proof. Well-definedness. Suppose $p = n\omega = n'\omega'$ with $n, n' \in N$ and $\omega, \omega' \in \Omega$. Write $\omega = \mathrm{diag}(k, 1)u_Q$ and $\omega' = \mathrm{diag}(k', 1)u_Q$ with $k, k' \in K_n$. Then $n\,\mathrm{diag}(k, 1)u_Q = n'\,\mathrm{diag}(k', 1)u_Q$, so right-multiplying by u_Q^{-1} gives

$$n\,\mathrm{diag}(k, 1) = n'\,\mathrm{diag}(k', 1).$$

Taking upper-left blocks shows that $N_n k = N_n k'$ in $N_n\backslash \mathrm{GL}_n(F)$. Since $k, k' \in K_n$, this implies $k' = uk$ for some $u \in N_n \cap K_n$. Hence $\mathrm{diag}(k', 1) = \mathrm{diag}(u, 1)\,\mathrm{diag}(k, 1)$ with $\mathrm{diag}(u, 1) \in N$, and therefore

$$n'^{-1}n = \mathrm{diag}(u, 1) \in N.$$

By the choice of K_n, we have $u \in N_n \cap K_n \subset \ker(\psi)$, hence $\psi(u) = 1$ and so $\psi^{-1}(n) = \psi^{-1}(n')$. Thus the value $\psi^{-1}(n)\lambda(v)^{-1}$ does not depend on the chosen decomposition $p = n\omega$.

Equivariance and support. If $m \in N$ and $p = n\omega \in N\Omega$, then $mp = (mn)\omega \in N\Omega$ and

$$\widetilde{f}(mp) = \psi^{-1}(mn)\lambda(v)^{-1} = \psi^{-1}(m)\,\widetilde{f}(p).$$

If $p \notin N\Omega$, then $mp \notin N\Omega$ as well (since $N\Omega$ is left N-stable), so $\widetilde{f}(mp) = 0 = \psi^{-1}(m)\widetilde{f}(p)$. Thus $\widetilde{f} \in \mathrm{Ind}_N^P(\psi^{-1})$. Moreover $\mathrm{supp}(\widetilde{f}) \subset N\Omega$ by definition, and its image in $N\backslash P$ is exactly $\overline{\Omega}$.

Values on Ω. If $k \in K_n$, then $\mathrm{diag}(k, 1)u_Q \in \Omega$, so taking $n = 1$ gives $\widetilde{f}(\mathrm{diag}(k, 1)u_Q) = \lambda(v)^{-1}$. □

5.3 Choose $W \in \mathcal{W}(\Pi, \psi^{-1})$ matching $\widetilde{f}$ on the mirabolic

By Lemma 1, choose $W \in \mathcal{W}(\Pi, \psi^{-1})$ such that

$$W(p) = \widetilde{f}(p) \qquad \text{for all } p \in P. \tag{4}$$

Lemma 5 (Explicit values and support of W on P)**.** *With W chosen by (4) we have:*

1. *$W(p) = 0$ for all $p \in P$ whose class in $N\backslash P$ lies outside $\overline{\Omega}$ (equivalently, $p \notin N\Omega$);*
2. *for all $k \in K_n$, $W(\mathrm{diag}(k, 1)u_Q) = \lambda(v)^{-1}$.*

Proof. Immediate from (4) and Lemma 4. □

6 Support and descent to the quotient

Corollary 6. *For $g \in \mathrm{GL}_n(F)$,*

$$W(\mathrm{diag}(g, 1)u_Q) \neq 0 \quad \Longleftrightarrow \quad N_n g \in U = (N_n \cap K_n)\backslash K_n.$$

Proof. By (3) and Lemma 5(1), $W(\mathrm{diag}(g, 1)u_Q) \neq 0$ iff the class of $\mathrm{diag}(g, 1)u_Q$ in $N\backslash P$ lies in $\overline{\Omega}$. By Lemma 3 this is equivalent to $N_n g \in U$. □

Lemma 7 (Well-definedness on the quotient)**.** *For each $s \in \mathbb{C}$, the function*

$$F_s(g) := W(\mathrm{diag}(g,1)u_Q)\, V(g)\, |\det g|^{s-\frac{1}{2}}$$

is left $N_n(F)$-invariant. Hence the integrand in (2) *is well-defined as a function on $N_n \backslash \mathrm{GL}_n(F)$.*

Proof. Let $u \in N_n(F)$. Since $\mathrm{diag}(u,1) \in N \subset P$, Whittaker equivariance gives

$$W(\mathrm{diag}(ug,1)u_Q) = W(\mathrm{diag}(u,1)\,\mathrm{diag}(g,1)u_Q) = \psi^{-1}(u)\, W(\mathrm{diag}(g,1)u_Q).$$

On the other hand, $V(ug) = \psi(u)V(g)$ and $\det(ug) = \det(g)$. Multiplying, the $\psi(u)$ and $\psi^{-1}(u)$ factors cancel, so $F_s(ug) = F_s(g)$. $\square$

7 Evaluation of the integral and the volume computation

By Corollary 6, the integrand in (2) is supported on the compact set U. Hence the integral converges absolutely for all $s \in \mathbb{C}$.

On U we may represent the class of g by some $k \in K_n$ (by Lemma 7, the value of the integrand does not depend on the representative). Then, using Lemma 5(2), right K_n–invariance of V, and $|\det k| = 1$, we get

$$W(\mathrm{diag}(g,1)u_Q)\, V(g)\, |\det g|^{s-\frac{1}{2}} = W(\mathrm{diag}(k,1)u_Q)\, V(k)\, |\det k|^{s-\frac{1}{2}} = \lambda(v)^{-1} \cdot \lambda(v) \cdot 1 = 1.$$

Therefore

$$I(s,W,V) = \int_U 1\, dg = \mathrm{vol}(U), \tag{5}$$

independent of s.

Deriving $\mathrm{vol}(U) = \mathrm{vol}(K_n)/\,\mathrm{vol}(N_n \cap K_n)$**.** Apply (1) to $\Phi = \mathbf{1}_{K_n} \in C_c^\infty(\mathrm{GL}_n(F))$. Then the left-hand side equals $\mathrm{vol}(K_n)$. For a fixed coset $g \in N_n \backslash \mathrm{GL}_n(F)$, consider the inner integral

$$\int_{N_n(F)} \mathbf{1}_{K_n}(u\tilde{g})\, du.$$

If $g \in U$, choose a representative $\tilde{g} = u_0 k$ with $u_0 \in N_n$ and $k \in K_n$. Then

$$\{u \in N_n :\ u\tilde{g} \in K_n\} = \{u \in N_n :\ uu_0k \in K_n\} = \{u \in N_n :\ uu_0 \in K_n\} = (N_n \cap K_n)\, u_0^{-1},$$

a left translate of $N_n \cap K_n$. Hence the inner integral equals $\mathrm{vol}(N_n \cap K_n)$. If $g \notin U$, then $N_n\tilde{g} \cap K_n = \varnothing$, so the inner integral is 0. Therefore (1) gives

$$\mathrm{vol}(K_n) = \int_{N_n\backslash \mathrm{GL}_n(F)} \left(\int_{N_n(F)} \mathbf{1}_{K_n}(u\tilde{g})\, du \right) dg = \mathrm{vol}(N_n \cap K_n) \int_U 1\, dg = \mathrm{vol}(N_n \cap K_n)\ \mathrm{vol}(U),$$

so

$$\mathrm{vol}(U) = \frac{\mathrm{vol}(K_n)}{\mathrm{vol}(N_n \cap K_n)} \in (0,\infty). \tag{6}$$

Combining (5) and (6) yields $I(s,W,V) \neq 0$ for all s.

8 Conclusion

We have constructed $V \in \mathcal{W}(\pi, \psi)$ and $W \in \mathcal{W}(\Pi, \psi^{-1})$ such that $I(s, W, V) = \mathrm{vol}(U) \in (0, \infty)$ for all $s \in \mathbb{C}$. Therefore the answer is **Yes**.

Remark 2. The argument did not use any special property of Q beyond $Q \in F^{\times}$. In particular it applies to the choice of Q specified in the problem statement (e.g. a generator of $\mathfrak{q}^{-1}$).