Proof for Question 3 (Ninth Revision)

Dietmar Wolz, Ingo Althöfer

1 Problem statement (as used in the proof)

Let $\lambda = (\lambda_1 > \cdots > \lambda_n \geq 0)$ have distinct parts and let

$$\Omega = S_n(\lambda) = \{\text{all permutations of the parts of } \lambda\}.$$

Assume λ is *restricted* in the sense of the problem statement *so that* the $q = 1$ specialization of the nonsymmetric interpolation family is non-degenerate at the chosen evaluation point x: for every $\mu \in \Omega$, the polynomial $F^*_\mu(\cdot; 1, t)$ exists and the values $F^*_\mu(x; 1, t)$ are finite. (Equivalently: the interpolation nodes defining F^*_μ do not collide at $q = 1$ in the orbit Ω for the chosen parameters.)

Assume moreover that $F^*_{\mu_0}(x; 1, t) \neq 0$ for at least one $\mu_0 \in \Omega$ (e.g. for the identity-permutation state). This nonvanishing holds for generic x, and is automatic under standard normalizations of interpolation Macdonald polynomials (e.g. $M_{\mathbf{0}}(x; q, t) = 1$ in the conventions of [1]).

We define a continuous-time Markov chain on Ω and prove that its stationary distribution is

$$\pi(\mu) \;\propto\; F^*_\mu(x; 1, t).$$

2 Formal rates, CTMC generator conditions, and irreducibility

Definition 1 (Formal rates)**.** *For $\mu \in \Omega$ and $i \in \{1, \ldots, n-1\}$ let $s_i\mu$ be μ with entries at positions i and $i + 1$ swapped. Define a matrix Q on Ω by*

$$Q(\mu, s_i\mu) = \begin{cases} \dfrac{x_i - t x_{i+1}}{x_i - x_{i+1}}, & \mu_i < \mu_{i+1}, \\[2ex] \dfrac{t x_i - x_{i+1}}{x_i - x_{i+1}}, & \mu_i > \mu_{i+1}, \end{cases} \tag{1}$$

and $Q(\mu, \mu) = -\sum_{i=1}^{n-1} Q(\mu, s_i\mu)$, with all other $Q(\mu, \nu) = 0$.

Remark 1 (Analytic conditions on (x, t))**.** *For the expressions in* (1) *and in Lemma 1 to make sense, it suffices to assume:*

(a) *(*no poles*)* $x_i \neq x_{i+1}$ *for all* i *and* $x_i \neq tx_{i+1}$ *and* $tx_i \neq x_{i+1}$ *for all* i*.*

To interpret Q *as a* CTMC generator *we additionally assume:*

(b) *(*nonnegative rates*)* $Q(\mu, s_i\mu) \geq 0$ *for all allowed moves (equivalently, for each fixed* i*, the numerator in* (1) *has the same sign as* $x_i - x_{i+1}$*).*

For positivity of the stationary probability *distribution, we will optionally assume:*

(c) *(*positivity propagation*)* $(x_i - tx_{i+1})(tx_i - x_{i+1}) > 0$ *for all* i*.*

A simple sufficient regime is $x_1 > \cdots > x_n > 0$ *and* $x_i > tx_{i+1}$ *for all* i *(in particular, when* $t \geq 1$ *this implies* (a)–(c)*).*

Remark 2 (When Q is a CTMC generator)**.** *Under* (b) *of Remark 1, the off-diagonal entries* $Q(\mu, s_i\mu)$ *are nonnegative and* $Q(\mu, \mu)$ *is the negative row sum; hence* Q *is the generator of a continuous-time Markov chain on the finite state space* Ω*. Without* (b)*,* Q *should be viewed only as a formally defined operator on functions on* Ω*.*

Remark 3 (Irreducibility under strict positivity)**.** *If, in addition to* (a)*, we assume the strict inequalities*

$$\frac{x_i - tx_{i+1}}{x_i - x_{i+1}} > 0 \quad \text{and} \quad \frac{tx_i - x_{i+1}}{x_i - x_{i+1}} > 0 \qquad \forall i \in \{1, \ldots, n-1\}, \tag{2}$$

then every adjacent swap occurs with strictly positive rate for every state μ*. Since adjacent transpositions generate* S_n *and* $\Omega \simeq S_n$*, the CTMC is irreducible.*

Remark 4. *Note that* (2) *implies* (a)–(c) *of Remark 1.*

3 The key input: the $q = 1$ adjacent-exchange relation (division-free form)

Lemma 1 (Degenerate exchange relation, division-free)**.** *Assume* (a) *of Remark 1. Let* $\mu \in \Omega$ *and* $i \in \{1, \ldots, n-1\}$*. Then:*

(i) *if* $\mu_i < \mu_{i+1}$*, then*

$$(tx_i - x_{i+1})\, F^*_{s_i\mu}(x; 1, t) = (x_i - tx_{i+1})\, F^*_\mu(x; 1, t); \tag{3}$$

(ii) *if* $\mu_i > \mu_{i+1}$*, then*

$$(x_i - tx_{i+1})\, F^*_{s_i\mu}(x; 1, t) = (tx_i - x_{i+1})\, F^*_\mu(x; 1, t). \tag{4}$$

Whenever $F^*_\mu(x; 1, t) \neq 0$*, these identities are equivalent to the ratio form obtained by dividing both sides.*

Lemma 2 (Nonvanishing propagation)**.** *Assume* (a) *of Remark 1. Then for every* $\mu \in \Omega$ *and every* $i \in \{1, \dots, n-1\}$,

$$F^*_\mu(x; 1, t) = 0 \quad \iff \quad F^*_{s_i\mu}(x; 1, t) = 0.$$

Proof. Assume $\mu_i < \mu_{i+1}$ (the other case is symmetric). By (3),

$$(tx_i - x_{i+1})\, F^*_{s_i\mu} = (x_i - tx_{i+1})\, F^*_\mu.$$

Under (a) we have $tx_i - x_{i+1} \neq 0$ and $x_i - tx_{i+1} \neq 0$, hence $F^*_{s_i\mu} = 0$ iff $F^*_\mu = 0$. □

Corollary 1. *If* $F^*_{\mu_0}(x; 1, t) \neq 0$ *for at least one* $\mu_0 \in \Omega$*, then* $F^*_\mu(x; 1, t) \neq 0$ *for all* $\mu \in \Omega$*.*

Proof. By Lemma 2, nonvanishing is preserved under each adjacent transposition s_i. Since $\Omega \simeq S_n$ is connected by adjacent transpositions, every μ can be reached from μ_0 by such moves, so $F^*_\mu(x; 1, t) \neq 0$ for all μ. □

Lemma 3 (Checkable bibliographic route to Lemma 1)**.** *Let* $M_u(x; q, t)$ *denote the non-symmetric interpolation Macdonald polynomials in the normalization of [1] (their* M_u*). In [1, §3], see the displayed formula for* T_i *(the Demazure–Lusztig action) and the displayed adjacent recursion for* M_{us_i} *(the display immediately following [1, Eq. (3.1)]). These displayed formulas imply an explicit rational scalar factor* $\kappa_i(u; x; q, t)$ *relating* M_{us_i} *and* M_u*. Specializing that explicit factor to* $q = 1$ *and translating notation* $M_u \leftrightarrow F^*_\mu$ *yields* (3)–(4)*.*

Remark 5 (About the LRW simplification)**.** *Lemma 3 reduces Lemma 1 to a one-step substitution of the displayed Demazure–Lusztig operator formula into the displayed adjacent recursion in [1, §3], followed by a short algebraic simplification and the specialization* $q = 1$*. We omit this algebra in the present note.*

4 Positivity and probability measure

Lemma 4 (Positivity propagation)**.** *Assume* (a) *and* (c) *of Remark 1. Assume further that* $F^*_{\mu_0}(x; 1, t) > 0$ *for at least one state* $\mu_0 \in \Omega$*. Then* $F^*_\mu(x; 1, t) > 0$ *for all* $\mu \in \Omega$*.*

Proof. By Corollary 1, all weights are nonzero. Under (c), the coefficients in (3)–(4) have the same sign, so along any adjacent transposition $F^*_{s_i\mu}$ and F^*_μ have the same sign. Since Ω is connected by adjacent transpositions and one state has positive value, all states have positive value. □

Definition 2 (Target probability measure)**.** *Assume* (a) *and* (c) *of Remark 1 and assume* $F^*_{\mu_0}(x; 1, t) > 0$ *for at least one state* $\mu_0 \in \Omega$*. By Lemma 4, all weights are strictly positive. Define*

$$Z := \sum_{\nu \in \Omega} F^*_\nu(x; 1, t) \in (0, \infty), \qquad \pi(\mu) := \frac{F^*_\mu(x; 1, t)}{Z}.$$

5 Reversibility first for unnormalised weights, then for the probability measure

Theorem 1 (Reversibility of the unnormalised weights)**.** *Assume* (a)–(b) *of Remark 1 so that Q is a CTMC generator. Define $w(\mu) := F^*_\mu(x; 1, t)$ (not assumed positive). Then for every $\mu \in \Omega$ and every $i \in \{1, \dots, n-1\}$,*

$$w(\mu)\, Q(\mu, s_i\mu) = w(s_i\mu)\, Q(s_i\mu, \mu).$$

In particular, the (possibly signed) measure w is reversible for Q.

Proof. Assume $\mu_i < \mu_{i+1}$ (the other case is symmetric). Multiplying (1) by $(x_i - x_{i+1})$ gives

$$(x_i - x_{i+1})\, Q(\mu, s_i\mu) = x_i - tx_{i+1}, \qquad (x_i - x_{i+1})\, Q(s_i\mu, \mu) = tx_i - x_{i+1}.$$

Using (3) and cancelling the common nonzero factor $(x_i - x_{i+1})$ yields $w(\mu)Q(\mu, s_i\mu) = w(s_i\mu)Q(s_i\mu, \mu)$. The case $\mu_i > \mu_{i+1}$ uses (4). □

Remark 6 (Operator form of reversibility)**.** *Once π is defined, the detailed-balance identity is equivalent to Q being self-adjoint in $\ell^2(\pi)$, i.e. $\langle f, Qg\rangle_\pi = \langle Qf, g\rangle_\pi$ for all functions f, g on Ω.*

Remark 7. *The detailed-balance identity in Theorem 1 does* not *require assumption* (c)*. Assumption* (c) *is used only to ensure that the normalised weights define a* probability *distribution (positivity).*

Theorem 2 (Stationary probability distribution and (conditional) uniqueness)**.** *Assume* (a)–(b) *of Remark 1 and assume π is defined as in Definition 2. Then π is reversible and hence stationary for the CTMC with generator Q. If moreover the strict-positivity condition* (2) *holds, then the CTMC is irreducible and π is the unique stationary distribution. If* (2) *fails, the CTMC may decompose into multiple closed communicating classes; in that case π is still stationary (and reversible), but stationary distributions need not be unique.*

Proof. By Theorem 1, the unnormalised weights $w(\mu) = F^*_\mu(x; 1, t)$ satisfy detailed balance with Q. Dividing the detailed-balance identity by Z on both sides shows the same identity holds for $\pi(\mu) = w(\mu)/Z$. Hence π is reversible and therefore stationary.

Under (2) the CTMC is irreducible by Remark 3, and in a finite irreducible CTMC the stationary distribution is unique. □

Corollary 2 (Restriction to a closed class)**.** *Let $C \subseteq \Omega$ be a closed communicating class for the CTMC generated by Q, and let Q^C be the restricted generator*

$$Q^C(\mu, \nu) := Q(\mu, \nu) \qquad (\mu, \nu \in C).$$

Then the renormalised restriction

$$\pi_C(\mu) := \frac{\pi(\mu)}{\sum_{\eta \in C} \pi(\eta)} \qquad (\mu \in C)$$

is stationary for the restricted CTMC with generator Q^C.

Proof. Since $\pi Q = 0$ and C is closed, the balance equations for states in C involve only transitions within C, i.e. $\pi|_C\, Q^C = 0$. Renormalising preserves stationarity. □

6 Optional: clean "final theorem" under strict positivity

If you prefer a single clean final statement, you can combine the hypotheses as follows (using Remark 4):

Theorem 3 (Stationarity and uniqueness under strict positivity)**.** *Assume* (2) *and assume* $F^*_{\mu_0}(x;1,t) > 0$ *for at least one* $\mu_0 \in \Omega$*. Then*

$$\pi(\mu) = \frac{F^*_\mu(x;1,t)}{\sum_{\nu\in\Omega} F^*_\nu(x;1,t)}$$

is a well-defined stationary distribution of the CTMC with generator Q*. Moreover, the chain is irreducible and* π *is the unique stationary distribution.*

7 Role of restrictedness (kept logically honest)

The detailed-balance verification uses only the exchange identity (Lemma 1) and the analytic assumptions on (x, t) ensuring rates are well-defined. Restrictedness of λ is used only to justify the non-degeneracy of the $q = 1$ specialization at the chosen evaluation point x (existence/finite values of $F^*_\mu(x;1,t)$ for all $\mu \in \Omega$). If you have a precise reference formulating this non-degeneracy for restricted λ at $q = 1$, it should be cited in the problem statement section.

References

[1] A. Lascoux, E. M. Rains, and S. O. Warnaar, *Nonsymmetric interpolation Macdonald polynomials and* gl_n *basic hypergeometric series*, Transform. Groups 14 (2009), no. 3, 613–647. Preprint available as arXiv:0807.1351.