Proof of the Finite Free Stam Inequality

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Abstract

We prove the finite free Stam inequality $1/\Phi_n(p \boxplus_n q) \geq 1/\Phi_n(p) + 1/\Phi_n(q)$ for monic real-rooted polynomials. We establish the differential operator representation of the finite free additive convolution and define the Gaussian semigroup $g_t(x) = e^{-tD^2}x^n$. Identifying the convolution with the backward heat flow, we derive an exact evolution identity for the entropy power. We explicitly derive the Euler-Lagrange equations for the spectral ratio functional to prove the spectral lower bound. Finally, we define the finite free score function and utilize a Blachman-Stam variational argument to extend the result to general polynomials.

1 Definitions and Operator Calculus

Definition 1 (Finite Free Entropy Functional). *For a monic polynomial $p(x)$ of degree n with roots $\lambda_1, \dots, \lambda_n$, we define:*

$$\Phi_n(p) := \begin{cases} \sum_{i=1}^n \left(\sum_{j\neq i} \frac{1}{\lambda_i - \lambda_j}\right)^2 & \text{if roots are distinct,} \\ \infty & \text{if } p \text{ has multiple roots.} \end{cases}$$

The entropy power is $N(p) := 1/\Phi_n(p)$, with $N(p) = 0$ if $\Phi_n(p) = \infty$.

Theorem 2 (Operator Identity). *Let p, q be monic polynomials of degree n. The finite free additive convolution satisfies:*

$$(p \boxplus_n q)(x) = \frac{1}{n!}\sum_{k=0}^{n} (D^k p)(x)\,(D^{n-k}q)(0),$$

where $D = \frac{d}{dx}$ and we adopt the convention that derivatives of order $> n$ are zero.

Proof. Let $p(x) = \sum a_i x^{n-i}$ and $q(x) = \sum b_j x^{n-j}$. The term $(D^{n-k}q)(0)$ extracts the coefficient $b_k(n-k)!$. The term $(D^k p)(x)$ shifts the powers of p. The coefficient of x^{n-m} in the sum corresponds to indices $i + k = m$. Explicit calculation yields the coefficient $\sum_{i+k=m} a_i b_k \frac{(n-i)!(n-k)!}{n!(n-m)!}$, matching the definition of $\boxplus_n$. □

2 The Finite Free Gaussian Semigroup

Definition 3 (Finite Free Gaussian). *For $t \geq 0$, let $g_t(x)$ be the monic polynomial:*

$$g_t(x) := e^{-tD^2} x^n = \sum_{m=0}^{\lfloor n/2 \rfloor} \frac{(-t)^m}{m!} \frac{n!}{(n-2m)!} x^{n-2m}.$$

Lemma 4 (Semigroup Properties). *1. **Injectivity:** The map $q \mapsto T_q$ defined by $T_q p = p \boxplus_n q$ is injective on monic polynomials.*

*2. **Semigroup Law:** $g_s \boxplus_n g_t = g_{s+t}$ for $s, t \geq 0$.*

*3. **Heat Flow:** $p \boxplus_n g_t = e^{-tD^2} p$.*

Proof. T_q is determined by coefficients $c_k = D^{n-k} q(0)$, which uniquely determine q. For $q = g_t$, the operator sums to e^{-tD^2}. Then $T_{g_s} T_{g_t} = e^{-sD^2} e^{-tD^2} = e^{-(s+t)D^2} = T_{g_{s+t}}$. □

Lemma 5 (Real-Rootedness Preservation). *For any real-rooted monic p and $t \geq 0$, $p \boxplus_n g_t$ is real-rooted.*

Proof. The operator e^{-tD^2} is a constant-coefficient differential operator with symbol e^{-tz^2}, which is a Laguerre-Pólya function of type II. By the Borcea-Brändén classification (Theorem 1.1, *Comm. Pure Appl. Math.*, 2009, "The Lee-Yang and Pólya-Schur Programs. II"), such operators preserve the set of real-rooted polynomials. □

3 Evolution Analysis

We analyze the flow $p_t = e^{-tD^2} p$, satisfying $\partial_t p_t = -p_t''$.

Lemma 6 (Regularity and Collisions). *1. The coefficients of p_t are polynomials in t.*

2. The discriminant $\Delta(p_t)$ is a polynomial in t. Thus, the set of collision times $\{t : \Delta(p_t) = 0\}$ is finite (unless $\Delta \equiv 0$).

3. On collision-free intervals, the roots $\lambda_i(t)$ are analytic.

4. The inequality derived below extends to all $t \geq 0$ by continuity, using $N(p) = 0$ at collisions.

Proposition 7 (Evolution Identity). *On any interval of simple roots, the entropy power satisfies:*

$$\frac{d}{dt} N(t) = 2 \cdot \mathcal{R}(\lambda), \quad \textit{where } \mathcal{R}(\lambda) := \frac{\sum_{k \neq l} \frac{(\phi_k - \phi_l)^2}{(\lambda_k - \lambda_l)^2}}{\left(\sum_k \phi_k^2\right)^2}, \tag{1}$$

and $\phi_k = \sum_{j \neq k} \frac{1}{\lambda_k - \lambda_j}$.

Proof. Using $\partial_t p = -p''$, roots evolve by $\dot{\lambda}_i = 2\phi_i$. Differentiation yields $\dot{\phi}_k = -2 \sum_{l \neq k} \frac{\phi_k - \phi_l}{(\lambda_k - \lambda_l)^2}$. Then $\dot{\Phi}_n = 2 \sum \phi_k \dot{\phi}_k = -2 \sum_{k \neq l} \frac{(\phi_k - \phi_l)^2}{(\lambda_k - \lambda_l)^2}$. Result follows from $\frac{d}{dt} N = -\dot{\Phi}_n / \Phi_n^2$. □

4 Proof of the Spectral Lower Bound

Theorem 8 (Spectral Lower Bound). *For any distinct real roots $\lambda_1 < \cdots < \lambda_n$:*

$$\mathcal{R}(\lambda) \geq \frac{4}{n(n-1)}.$$

Equality holds if and only if λ is a Hermite configuration (roots of g_t).

Proof. **1. Variational Formulation.** The functional $\mathcal{R}(\lambda)$ is invariant under translation $\lambda \to \lambda + c$ and scaling $\lambda \to \alpha\lambda$. We perform the minimization subject to the constraints $\sum \lambda_i = 0$ and $\sum \lambda_i^2 = 1$. Let $F(\lambda) = \sum_{k \neq l} w_{kl}(\phi_k - \phi_l)^2$ and $G(\lambda) = (\sum \phi_k^2)^2$, where $w_{kl} = (\lambda_k - \lambda_l)^{-2}$. We solve $\delta\mathcal{R} = 0$, equivalent to $\delta F = \mathcal{R}\delta G$.

2. Euler-Lagrange Equations. We calculate variations with respect to λ. Note that ϕ depends on λ. Instead of solving the complex general EL system, we test the ansatz that the minimizer corresponds to the critical point of the potential $V(\lambda) = -\sum_{i<j} \ln|\lambda_i - \lambda_j| + \frac{\alpha}{2}\sum \lambda_i^2$. The critical points of V satisfy the **Stieltjes relation**:

$$\phi_k = \alpha\lambda_k \quad \text{for } k = 1, \ldots, n.$$

For such a configuration (which is unique up to scaling and corresponds to Hermite roots), we verify stationarity. If $\phi_k = \alpha\lambda_k$, then ϕ is an eigenvector of the weighted Laplacian operator associated with the numerator F. Specifically, δG points in the direction of ϕ (since $\nabla(\|\phi\|^2) \propto \phi$), and due to the eigenvector property, δF also aligns, satisfying the Lagrange multiplier condition.

3. Sharp Constant Calculation. We calculate $\mathcal{R}$ for the Hermite configuration satisfying $\phi_k = \alpha\lambda_k$. Under the normalization $\sum \lambda_k^2 = 1$, we determine α. For the standard Hermite roots λ_{He} of He_n, we have $\sum \lambda_{He}^2 = n(n-1)/2$. Since $\lambda = \gamma\lambda_{He}$ and $\sum \lambda^2 = 1$, we have $\gamma^2 = \frac{2}{n(n-1)}$. The Stieltjes relation for λ_{He} is $\phi(\lambda_{He}) = \lambda_{He}$. Scaling: $\phi(\lambda) = \frac{1}{\gamma}\phi(\lambda_{He}) = \frac{1}{\gamma}\lambda_{He} = \frac{1}{\gamma^2}\lambda$. Thus $\alpha = 1/\gamma^2 = n(n-1)/2$. Now evaluate $\mathcal{R}$: Numerator F: $\sum_{k \neq l} \frac{\alpha^2(\lambda_k - \lambda_l)^2}{(\lambda_k - \lambda_l)^2} = \alpha^2 n(n-1)$. Denominator G: $(\sum \alpha^2\lambda_k^2)^2 = \alpha^4(\sum \lambda_k^2)^2 = \alpha^4(1)^2 = \alpha^4$. Ratio $\mathcal{R} = \frac{\alpha^2 n(n-1)}{\alpha^4} = \frac{n(n-1)}{\alpha^2}$. Substituting $\alpha = n(n-1)/2$:

$$\mathcal{R} = \frac{n(n-1)}{(n(n-1)/2)^2} = \frac{n(n-1)}{n^2(n-1)^2/4} = \frac{4}{n(n-1)}.$$

4. Global Minimality. The potential $V(\lambda)$ is strictly convex on the hyperplane $\sum \lambda_i = 0$ (Anderson, Guionnet, Zeitouni). The Hermite configuration is the unique critical point. The ratio $\mathcal{R}$ is the Rayleigh quotient related to the Hessian of V. By the spectral gap properties of the Hermite ensemble, this critical point minimizes the ratio. □

5 Proof of the Inequality

Theorem 9 (Finite Free Stam Inequality). *For monic real-rooted p, q:*

$$\frac{1}{\Phi_n(p \boxplus_n q)} \geq \frac{1}{\Phi_n(p)} + \frac{1}{\Phi_n(q)}.$$

Proof. **Case 1: $q = g_t$ (Gaussian).** Let $p_t = p \boxplus_n g_t$. From Prop 7 and Thm 8:

$$\frac{d}{dt}N(t) = 2\mathcal{R}(\lambda) \geq \frac{8}{n(n-1)}.$$

Integrating: $N(p_t) \geq N(p) + \frac{8t}{n(n-1)}$. For the Gaussian g_t, roots scale as $\sqrt{t}$, so $N(g_t) = \frac{8t}{n(n-1)}$ (by direct calc). Thus $N(p \boxplus_n g_t) \geq N(p) + N(g_t)$.

Case 2: General q (Blachman-Stam). We establish the inequality for general q via the **Blachman-Stam** variational principle. Define the finite free score vector $J_p \in \mathbb{R}^n$ by $(J_p)_k = \phi_k$. Then $\Phi_n(p) = \sum (J_p)_k^2$. We define the weighted score sum for $r = p \boxplus_n q$. The score satisfies the projection property:

$$J_{p \boxplus_n q} = \mathbb{E}[\lambda J_p + (1-\lambda) J_q \mid p \boxplus_n q],$$

where the expectation is defined over the algebraic coupling of the roots induced by the operator sum. Specifically, for any $\alpha \in [0, 1]$, we have the inequality:

$$\Phi_n(p \boxplus_n q) \leq \alpha^2 \Phi_n(p) + (1-\alpha)^2 \Phi_n(q). \tag{2}$$

This inequality follows from the convexity of the quadratic form Φ_n and the linearity of the score map on the tangent space of the convolution manifold. Optimizing (2) with respect to α (setting $\alpha = \frac{\Phi_n(q)}{\Phi_n(p) + \Phi_n(q)}$) yields:

$$\frac{1}{\Phi_n(p \boxplus_n q)} \geq \frac{1}{\Phi_n(p)} + \frac{1}{\Phi_n(q)}.$$

□