Characterization of the $\mathcal{O}$-Slice Filtration

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Abstract

We define the slice filtration on the category of G-spectra adapted to an incomplete transfer system $\mathcal{O}$ (equivalently, an indexing system $\mathcal{I}$). We state and prove a characterization of the $\mathcal{O}$-slice connectivity of a connective G-spectrum in terms of the connectivity of its geometric fixed points $\Phi^H(X)$.

1 Definitions and Setup

Let G be a finite group. An N_∞ operad $\mathcal{O}$ is equivalent to the data of an *indexing system* $\mathcal{I}$ [1]. For each subgroup $H \leq G$, $\mathcal{I}(H)$ is a collection of finite H-sets closed under finite coproducts, restriction to subgroups, and self-induction.

Definition 1.1 (Admissible Slice Cells)**.** For a subgroup $H \leq G$, let $T \in \mathcal{I}(H)$ be an admissible H-set. Let $V_T = \mathbb{R}[T]$ denote the corresponding permutation representation. An *$\mathcal{O}$-slice cell* is a G-spectrum of the form

$$G_+ \wedge_H S^{V_T}, \quad H \leq G,\ T \in \mathcal{I}(H).$$

We assign this cell a filtration degree equal to the cardinality $|T|$.

Definition 1.2 ($\mathcal{O}$-Slice Filtration)**.** For $n \in \mathbb{Z}$, let $\tau^{\mathcal{O}}_{\geq n} \subseteq \mathrm{Sp}^G$ be the localizing subcategory generated by all $\mathcal{O}$-slice cells $G_+ \wedge_H S^{V_T}$ satisfying

$$|T| \geq n.$$

A G-spectrum X is *$\mathcal{O}$-slice n-connective* if $X \in \tau^{\mathcal{O}}_{\geq n}$.

Remark 1.3. The definition of the filtration degree as $|T|$ is a convention. Our explicit bound $\lambda_H(n)$ below is derived specifically for this choice.

Definition 1.4 (Connectivity Bound $\lambda_H(n)$)**.** For a subgroup $H \leq G$ and integer n, define the bound $\lambda_H(n)$ as:

$$\lambda_H(n) = \min \{ |T/H| \mid T \in \mathcal{I}(H), |T| \geq n \}.$$

Here $|T/H|$ denotes the number of H-orbits in T, which corresponds to $\dim((\mathbb{R}[T])^H)$.

2 Lemmas

Lemma 2.1 (Restriction). *For any subgroup $K \leq G$, restriction preserves $\mathcal{O}$-slice connectivity:*

$$\mathrm{Res}^G_K(\tau^{\mathcal{O}}_{\geq n}) \subseteq \tau^{\mathcal{O}|_K}_{\geq n}.$$

Proof. Let $G_+ \wedge_H S^{V_T}$ be a generator with $T \in \mathcal{I}(H)$ and $|T| \geq n$. By the Mackey decomposition:

$$\mathrm{Res}^G_K(G_+ \wedge_H S^{V_T}) \simeq \bigvee_{g \in K\backslash G/H} K_+ \wedge_{K \cap {}^gH} \mathrm{Res}^H_{K \cap {}^gH}(S^{V_T}).$$

Let $M = K \cap {}^gH$. The restriction of V_T is the permutation representation of $\mathrm{Res}^H_M(T)$. By the closure properties of indexing systems, if $T \in \mathcal{I}(H)$, then $\mathrm{Res}^H_M(T) \in \mathcal{I}(M)$. Thus, each summand is an $\mathcal{O}|_K$-slice cell. The filtration degree is preserved because restriction of the action does not change the cardinality of the underlying set: $|\mathrm{Res}^H_M T| = |T| \geq n$. □

Lemma 2.2 (Induction Compatibility). *If $X \in \tau^{\mathcal{O}|_K}_{\geq n}$ for a subgroup $K \leq G$, then $\mathrm{Ind}^G_K X \in \tau^{\mathcal{O}}_{\geq n}$.*

Proof. It suffices to show this for generators. Let $K_+ \wedge_H S^{V_T}$ be a generator for $\tau^{\mathcal{O}|_K}_{\geq n}$, where $H \leq K$ and $T \in \mathcal{I}(H)$. Then

$$\mathrm{Ind}^G_K(K_+ \wedge_H S^{V_T}) \simeq G_+ \wedge_K (K_+ \wedge_H S^{V_T}) \simeq G_+ \wedge_H S^{V_T}.$$

Since $T \in \mathcal{I}(H)$, this is a valid generator for $\tau^{\mathcal{O}}_{\geq n}$ with dimension $|T|$. (Note: The self-induction axiom of indexing systems ensures that we can formally regard this as induced from the K-set $K \times_H T \in \mathcal{I}(K)$, ensuring compatibility with the structure of $\mathcal{I}$ on K.) □

Lemma 2.3 (Proper Isotropy Reduction). *Let $\mathcal{P}$ be the family of proper subgroups of G. For a connective G-spectrum X,*

$$E\mathcal{P}_+ \wedge X \in \tau^{\mathcal{O}}_{\geq n} \iff \mathrm{Res}^G_K(X) \in \tau^{\mathcal{O}|_K}_{\geq n} \quad \textit{for all } K < G.$$

Proof. The space $E\mathcal{P}$ is a G-CW complex built from cells $G/K \times D^m$ where $K \in \mathcal{P}$. Thus, $E\mathcal{P}_+ \wedge X$ lies in the localizing subcategory generated by $G/K_+ \wedge X \simeq \mathrm{Ind}^G_K \mathrm{Res}^G_K X$ for $K < G$. If $\mathrm{Res}^G_K X \in \tau^{\mathcal{O}|_K}_{\geq n}$, then by Lemma 2.2, the induced term is in $\tau^{\mathcal{O}}_{\geq n}$. Since $\tau^{\mathcal{O}}_{\geq n}$ is localizing, $E\mathcal{P}_+ \wedge X \in \tau^{\mathcal{O}}_{\geq n}$.

Conversely, suppose $E\mathcal{P}_+ \wedge X \in \tau^{\mathcal{O}}_{\geq n}$. Let K be a proper subgroup. For any subgroup $J \leq K$, J is also a proper subgroup of G (since $K < G$). Therefore, the fixed point space $(E\mathcal{P})^J$ is contractible for all $J \leq K$. This implies that the restriction $E\mathcal{P}|_K$ is K-equivariantly contractible. Consequently, $\mathrm{Res}^G_K(E\mathcal{P}_+) \simeq S^0$, and

$$\mathrm{Res}^G_K(E\mathcal{P}_+ \wedge X) \simeq \mathrm{Res}^G_K(E\mathcal{P}_+) \wedge \mathrm{Res}^G_K X \simeq S^0 \wedge \mathrm{Res}^G_K X \simeq \mathrm{Res}^G_K X.$$

By Lemma 2.1, the restriction of any element in $\tau^{\mathcal{O}}_{\geq n}$ lies in $\tau^{\mathcal{O}|_K}_{\geq n}$. Therefore, $\mathrm{Res}^G_K X \in \tau^{\mathcal{O}|_K}_{\geq n}$. □

Lemma 2.4 (Geometric Fixed Points of Generators). *For a generator $C = G_+ \wedge_H S^{V_T}$ with $T \in \mathcal{I}(H)$, the geometric fixed points at G are:*

$$\Phi^G(C) \simeq \begin{cases} S^{(V_T)^G} & \textit{if } H = G \\ * & \textit{if } H < G \end{cases}$$

The connectivity of $\Phi^G(S^{V_T})$ is exactly $\dim((V_T)^G) = |T/G|$ *(the number of orbits).*

Proof. If $H < G$, the generator is induced from a proper subgroup, so its geometric fixed points vanish (as $\Phi^G \circ \mathrm{Ind}_H^G \simeq *$). If $H = G$, Φ^G is monoidal and $\Phi^G(S^V) \cong S^{V^G}$. For a permutation representation $\mathbb{R}[T]$, the fixed subspace is spanned by the orbit sums, so its dimension is $|T/G|$. $\square$

3 Main Theorem

Theorem 3.1. *Let G be a finite group and $\mathcal{O}$ an N_∞ operad with indexing system $\mathcal{I}$. A connective G-spectrum X is $\mathcal{O}$-slice n-connective ($X \in \tau_{\geq n}^{\mathcal{O}}$) if and only if for all $H \leq G$, the geometric fixed points $\Phi^H(X)$ are $\lambda_H(n)$-connective.*

Proof. We proceed by induction on the order of G. The base case $|G| = 1$ is immediate as $\lambda_1(n) = n$ and $\tau_{\geq n}^{\mathcal{O}}$ is the standard Postnikov filtration. Assume the theorem holds for all proper subgroups of G.

Step 1: Isotropy Separation. Consider the isotropy separation sequence:

$$E\mathcal{P}_+ \wedge X \longrightarrow X \longrightarrow \widetilde{E}\mathcal{P} \wedge X.$$

Since $\tau_{\geq n}^{\mathcal{O}}$ is a localizing subcategory, $X \in \tau_{\geq n}^{\mathcal{O}}$ if and only if both terms are in $\tau_{\geq n}^{\mathcal{O}}$.

Step 2: The Proper Part. By Lemma 2.3, $E\mathcal{P}_+ \wedge X \in \tau_{\geq n}^{\mathcal{O}}$ is equivalent to $\mathrm{Res}_K^G X \in \tau_{\geq n}^{\mathcal{O}|_K}$ for all proper $K < G$. Applying the inductive hypothesis to K, this holds if and only if for all $L \leq K$, $\Phi^L(\mathrm{Res}_K^G X) \simeq \Phi^L(X)$ is $\lambda_L(n)$-connective. As K ranges over all proper subgroups, this condition is equivalent to $\Phi^H(X)$ being $\lambda_H(n)$-connective for all proper $H < G$.

Step 3: The Geometric Part. The term $\widetilde{E}\mathcal{P} \wedge X$ is geometric (trivial restriction to proper subgroups). Since smashing with $\widetilde{E}\mathcal{P}$ kills any generator induced from a proper subgroup (recall $\Phi^G(\mathrm{Ind}_H^G Y) \simeq *$ for $H < G$), $\widetilde{E}\mathcal{P} \wedge X \in \tau_{\geq n}^{\mathcal{O}}$ if and only if it lies in the localizing subcategory generated by cells S^{V_T} with $T \in \mathcal{I}(G)$ and $|T| \geq n$.

For geometric spectra, the functor Φ^G is conservative (detects equivalences) and preserves connectivity [2]. Thus, the condition is equivalent to $\Phi^G(\widetilde{E}\mathcal{P} \wedge X) \simeq \Phi^G(X)$ lying in the localizing subcategory of spectra generated by spheres $S^{V_T^G}$ where $|T| \geq n$. It is a standard result that the localizing subcategory generated by spheres of dimension $\geq k$ is precisely the category of k-connective spectra. Therefore, the condition is that $\Phi^G(X)$ is k-connective where

$$k = \min\{\dim(V_T^G) \mid T \in \mathcal{I}(G), |T| \geq n\} = \min\{|T/G| \mid T \in \mathcal{I}(G), |T| \geq n\} = \lambda_G(n).$$

Conclusion. Combining Step 2 (conditions for $H < G$) and Step 3 (condition for $H = G$), $X \in \tau_{\geq n}^{\mathcal{O}}$ if and only if $\Phi^H(X)$ is $\lambda_H(n)$-connective for all $H \leq G$. $\square$

References

[1] A. J. Blumberg and M. A. Hill. N_∞ operads and the multiplicative norm. *Geometry & Topology*, 19(6):3683–3735, 2015.

[2] M. A. Hill, M. J. Hopkins, and D. C. Ravenel. On the non-existence of elements of Kervaire invariant one. *Annals of Mathematics*, 184(1):1–262, 2016.