Light Sets of Vertices with an Improved Universal Constant (A polished "Spielman-style" note for First Proof Q6)

ChatGpt 5.2 Extended Thinking

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Abstract

Spielman proved that every weighted graph on n vertices contains an ε-light set of size at least $\varepsilon n/42$ for every $0 < \varepsilon < 1$. In this note we show that the same proof yields a slightly better constant:

$$|S| \geq \frac{5}{206} \varepsilon n \qquad (\approx \varepsilon n/41.2),$$

by a strict parameter re-tuning only (no change in the argument).

1   Setup

Let $G = (V, E, w)$ be a weighted graph with $|V| = n$ and Laplacian

$$L = \sum_{(s,t) \in E} w(s,t)\,(\delta_s - \delta_t)(\delta_s - \delta_t)^{\mathsf{T}}.$$

For a vertex set $S \subseteq V$ let G_S denote the subgraph with vertex set V containing only edges with both endpoints in S, and let L_S be its Laplacian (so L_S has the same dimension as L).

Definition 1 (ε-light set)**.** *For $0 < \varepsilon < 1$, a set $S \subseteq V$ is ε-light if*

$$\varepsilon L \succeq L_S.$$

It is convenient (as in Spielman) to work with the normalized matrix

$$\widetilde{L}_S := L^{\dagger/2} L_S L^{\dagger/2},$$

where $L^{\dagger}$ denotes the Moore–Penrose pseudoinverse. Since $L^{\dagger/2} L L^{\dagger/2}$ is the orthogonal projector onto $\operatorname{im}(L)$, the condition $\varepsilon L \succeq L_S$ is equivalent to $\|\widetilde{L}_S\| \leq \varepsilon$ on $\operatorname{im}(L)$.

2   Main lemma with improved constant

Lemma 1 (Improved light-set bound)**.** *For every weighted graph $G = (V, E, w)$ with n vertices and every $0 < \varepsilon < 1$, there exists an ε-light set $S \subseteq V$ with*

$$|S| \geq \frac{5}{206} \varepsilon n.$$

Parameters

Fix the rational parameter

$$a := \frac{103}{5} = 20.6,$$

and define

$$\delta := \frac{a}{n}, \qquad \varphi := \frac{n}{a}, \qquad \sigma := \left\lfloor \frac{\varepsilon n}{2a} \right\rfloor = \left\lfloor \frac{5\varepsilon n}{206} \right\rfloor.$$

Note that $a > 20.5$, which is the only numeric requirement needed below.

3 Two invariants: leverage and barrier

3.1 Leverage scores

For an edge $(s, t) \in E$ define its leverage score

$$\ell(s, t) := w(s, t) \, (\delta_s - \delta_t)^{\mathsf{T}} L^\dagger (\delta_s - \delta_t).$$

If $(s, t) \notin E$ set $\ell(s, t) := 0$. For vertex sets S, T define

$$\ell(S, T) := \sum_{s \in S} \sum_{t \in T} \ell(s, t), \qquad \ell(S) := \ell(S, V \setminus S).$$

Spielman observes that $\ell(S, T) = \mathrm{Tr}(\widetilde{L}_{S,T})$, where $L_{S,T}$ is the Laplacian of the bipartite subgraph using edges between S and T.

3.2 Barrier potential

For a symmetric matrix A with eigenvalues $\lambda_1 \geq \cdots \geq \lambda_n$ and $u > \lambda_1$, define

$$\Phi_\sigma^u(A) := \sum_{i=1}^{\sigma} \frac{1}{u - \lambda_i}.$$

For a vertex set S define $\Phi_\sigma^u(S) := \Phi_\sigma^u(\widetilde{L}_S)$.

Our greedy construction maintains:

- **(L)** $\ell(S) \leq 4|S|$,

- **(B)** $\Phi_\sigma^u(S) \leq \varphi$,

while increasing u in steps of size δ.

4 The one-step extension lemma

Lemma 2 (One-step extension). *If $|S| \leq \sigma$, $\ell(S) \leq 4|S|$, and $\Phi_\sigma^u(S) \leq \varphi$, then there exists $t \notin S$ such that*

$$\ell(S \cup \{t\}) \leq \ell(S) + 4 \qquad and \qquad \Phi_\sigma^{u+\delta}(S \cup \{t\}) \leq \varphi.$$

Proof. This is identical to Spielman's Lemma 1.2. It combines:

- Lemma 3 below, which says *more than half* of $t \notin S$ satisfy the leverage increment bound.

- Lemma 4 below, which (under our parameter choice $a > 20.5$) says *at least half* of $t \notin S$ satisfy the barrier update.

Therefore some $t \notin S$ satisfies both. $\qquad\square$

5 Constructing the light set

Proof of Lemma 1. Set $u_0 := \varepsilon/2$ and start with a singleton $S_0 = \{v_0\}$. Since G_{S_0} has no edges, $\widetilde{L}_{S_0} = 0$ and

$$\Phi_\sigma^{u_0}(S_0) = \frac{\sigma}{u_0} \; \leq \; \frac{\varepsilon n/(2a)}{\varepsilon/2} = \frac{n}{a} = \varphi.$$

Also $\ell(S_0) = 0 \leq 4|S_0|$.

Applying Lemma 2 iteratively σ times constructs a set S with $|S| = \sigma + 1$ such that

$$\Phi_\sigma^{u_0 + \sigma\delta}(S) \leq \varphi \quad \text{and} \quad \ell(S) \leq 4\sigma.$$

In particular $\Phi_\sigma^{u_0+\sigma\delta}(S) < \infty$, so the largest eigenvalue of $\widetilde{L}_S$ is at most $u_0 + \sigma\delta$. By definition of δ and σ,

$$u_0 + \sigma\delta \ \leq \ \frac{\varepsilon}{2} + \frac{\varepsilon n}{2a} \cdot \frac{a}{n} = \varepsilon.$$

Thus $\|\widetilde{L}_S\| \leq \varepsilon$, which is equivalent to $\varepsilon L \succeq L_S$, i.e. S is ε-light.

Finally, $|S| = \sigma + 1 \geq \sigma \geq \frac{5}{206}\varepsilon n - 1$, and in particular $|S| \geq \frac{5}{206}\varepsilon n$ for all n after absorbing the $+1/\lfloor\cdot\rfloor$ rounding as in the official note. $\qquad\square$

6   The two "half-good" lemmas

We include the two selection lemmas in the same style as Spielman's note.

6.1   Leverage half-good

Lemma 3. *Let $S \subseteq V$. Then for more than half of $t \notin S$,*

$$\ell(S \cup \{t\}) \leq \ell(S) + 4.$$

Proof. For $t \notin S$,

$$\ell(S \cup \{t\}) = \ell(S \cup \{t\}, V \setminus (S \cup \{t\})) \leq \ell(S \cup \{t\}, V \setminus S) = \ell(S) + \ell(t, V \setminus S).$$

Thus it suffices to show that more than half of $t \notin S$ satisfy $\ell(t, V \setminus S) \leq 4$. This follows from nonnegativity of ℓ and Claim 1 applied to $T = V \setminus S$:

$$\sum_{t \in V \setminus S} \ell(t, V \setminus S) \leq 2(|V \setminus S| - 1) < 2|V \setminus S|.$$

So by Markov, fewer than half can exceed 4. $\qquad\square$

Claim 1. *For every $T \subseteq V$,*
$$\sum_{t \in T} \ell(t, T) \leq 2(|T| - 1).$$

Proof. This is Spielman's Claim 2.2. One writes $\sum_{t \in T} \ell(t, T) = 2\operatorname{Tr}(L_T L^\dagger)$, notes $L_T \preceq L$ so all eigenvalues of $L_T L^\dagger$ lie in $[0, 1]$, and that $\operatorname{rank}(L_T) \leq |T| - 1$, hence the trace is $< |T|$. $\qquad\square$

6.2   Barrier half-good (where the constant changes)

Spielman defines a nonnegative quantity $U(S, t)$ (a ratio of trace terms derived from the BSS barrier update). The detailed matrix inequalities proving the bound below are unchanged; the only change is the final numeric estimate.

Lemma 4. *If $|S| \leq \sigma$, $\Phi_\sigma^u(S) \leq \varphi$, and $\ell(S) \leq 4|S|$, then for at least half of $t \notin S$,*

$$U(S, t) < 1,$$

and consequently the barrier update $\Phi_\sigma^{u+\delta}(S \cup \{t\}) \leq \varphi$ holds for at least half of $t \notin S$.

Proof. Spielman's proof shows the summation bound

$$\sum_{t \notin S} U(S, t) \leq \frac{5}{\delta} + 5\varphi.$$

Since $U(S, t) \geq 0$, at least half of the $t \notin S$ satisfy $U(S, t) \leq \frac{2}{n - |S|} \left(\frac{5}{\delta} + 5\varphi \right)$. With our parameters $\delta = a/n$, $\varphi = n/a$ this becomes

$$\frac{2}{n - |S|} \left(\frac{5}{\delta} + 5\varphi \right) = \frac{2}{n - |S|} \cdot \frac{10n}{a}.$$

Using $|S| \leq \sigma \leq n/(2a)$ gives $n - |S| \geq \frac{(2a-1)n}{2a}$, hence

$$\frac{2}{n - |S|} \cdot \frac{10n}{a} \leq \frac{20n/a}{(2a - 1)n/(2a)} = \frac{40}{2a - 1}.$$

Because $a = 103/5 > 20.5$, we have $\frac{40}{2a-1} < 1$, so for at least half of $t \notin S$ we get $U(S, t) < 1$. $\quad \square$

Remark

The same note yields the general statement: for any real $a > 20.5$, choosing $\delta = a/n$, $\varphi = n/a$, and $\sigma = \lfloor \varepsilon n/(2a) \rfloor$ gives a universal constant $c = 1/(2a)$. The choice $a = 103/5$ is simply a convenient explicit improvement above $1/42$.