Milking the pages 31–34 proof for First Proof Q6: improving $c = \frac{1}{256}$ to $c = \frac{1}{20}$

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Goal. For a graph $G = (V, E)$ with Laplacian L and $S \subseteq V$, let L_S be the Laplacian of the edge-induced subgraph $(V, E(S, S))$. A set S is *ε-light* if

$$\varepsilon L - L_S \succeq 0.$$

Pages 31–34 of the uploaded report prove the existence of an ε-light set of size at least $\varepsilon|V|/256$ via a one-sided BSS barrier method and a partial coloring procedure.[1] This note keeps *exactly the same proof* and only retunes the numerical parameters in Step 3/6.

Setup recalled from the report (very briefly)

Let $n := |V|$. Work on $\mathrm{range}(L) = (\ker L)^\perp$ and write $d := \mathrm{rank}(L) \le n$. For each edge $e = \{u, v\}$ define

$$L_e := (e_u - e_v)(e_u - e_v)^\top, \qquad A_e := L^{-1/2} L_e L^{-1/2} \succeq 0,$$

so that $\sum_{e \in E} A_e = I$ on $\mathrm{range}(L)$, and for any $S \subseteq V$,

$$L^{-1/2} L_S L^{-1/2} = \sum_{e \in E(S,S)} A_e \quad \text{on } \mathrm{range}(L).$$

Hence it suffices to find S with $\sum_{e \in E(S,S)} A_e \preceq \varepsilon I$.

The report colors vertices one by one using r colors. After t steps, with colored set T (size t), the matrix

$$M_t := \sum_{\substack{\{u,v\} \in E \\ u,v \in T \\ \mathrm{col}(u) = \mathrm{col}(v)}} A_{uv}$$

tracks the contribution of monochromatic edges among already-colored vertices. If we color a new vertex v with color γ, the increment is

$$B_v^\gamma := \sum_{\substack{u \in T \\ \mathrm{col}(u) = \gamma \\ \{u,v\} \in E}} A_{uv} \succeq 0, \qquad M_{t+1} = M_t + B_v^\gamma.$$

Using the one-sided BSS barrier lemma (Lemma 6.1 in the report), the report shows that if

$$\frac{d/u_0}{mr} + \frac{1}{\delta m r} < 1 \tag{1}$$

holds for all $t < k$ (where $m = n - t$), then one can choose (v, γ) at each step so that the barrier invariant propagates and finally $M_k \preceq u_k I$ with $u_k := u_0 + k\delta$.

[1] In the report, this is Section 6, equations (31)–(42), and Steps 1–6.

Parameter retuning

We modify only Step 3/6 (the choices in equation (36) of the report).

Theorem 1 (Milked constant)**.** *For every graph $G = (V, E)$ and every $\varepsilon \in (0, 1)$, there exists an ε-light set $S \subseteq V$ with*

$$|S| \;\geq\; \frac{\varepsilon}{20}\,|V|.$$

Proof. If $E = \varnothing$, then $L_S = 0$ for every S and we may take $S = V$. Hence assume $E \neq \varnothing$, so $d > 0$.

Step 3 (new choices). Set

$$r := \left\lceil \frac{4}{\varepsilon} \right\rceil, \qquad u_0 := \frac{2\varepsilon}{3}, \qquad \delta := \frac{\varepsilon}{n}, \qquad k := \left\lfloor \frac{n}{3} \right\rfloor.$$

We run the *same* partial coloring process as in the report for k steps and r colors.

Step 4 (barrier feasibility). For $t < k$ we have $m = n - t \geq n - k \geq \frac{2n}{3}$. Using $d \leq n$, we bound the average from the report (equation (42) there) as

$$\frac{d/u_0}{mr} + \frac{1}{\delta m r} \;\leq\; \frac{n}{(2\varepsilon/3)\cdot(2n/3)\cdot r} \;+\; \frac{1}{(\varepsilon/n)\cdot(2n/3)\cdot r} = \left(\frac{9}{4} + \frac{3}{2}\right)\frac{1}{\varepsilon r} = \frac{15}{4}\cdot\frac{1}{\varepsilon r}.$$

Since $r = \lceil 4/\varepsilon \rceil$, we have $\varepsilon r \geq 4$, hence the right-hand side is at most $15/16 < 1$. Thus condition (1) holds at every step $t < k$, so (exactly as in the report) we can choose a pair (v, γ) satisfying the one-sided barrier condition and propagate the invariant through all k steps.

Step 5 (extracting an ε-light class). After k steps, the colored set T is partitioned into color classes $S_1, \ldots, S_r$. As in the report,

$$M_k = \sum_{a=1}^{r} L^{-1/2} L_{S_a} L^{-1/2} \quad \text{on range}(L),$$

and each summand is PSD. The barrier invariant gives $M_k \preceq u_k I$ with $u_k = u_0 + k\delta \leq u_0 + \varepsilon/3 = \varepsilon$. Therefore each class satisfies

$$L^{-1/2} L_{S_a} L^{-1/2} \;\preceq\; M_k \;\preceq\; \varepsilon I,$$

and hence (by the normalization in the report's Step 1) each S_a is ε-light. Let S be the largest class, so $|S| \geq k/r$.

Step 6 (size lower bound). If $n \leq 11$, then the singleton set has $L_S = 0$ and is ε-light, and $1 \geq \varepsilon n/20$ holds because $\varepsilon \leq 1$ and $n/20 \leq 11/20 < 1$.

Assume now $n \geq 12$. Then $k = \lfloor n/3 \rfloor \geq n/4$. Also $r = \lceil 4/\varepsilon \rceil \leq 4/\varepsilon + 1 \leq 5/\varepsilon$ since $4/\varepsilon \geq 4$ for $\varepsilon \in (0, 1)$. Hence

$$|S| \;\geq\; \frac{k}{r} \;\geq\; \frac{n/4}{5/\varepsilon} \;=\; \frac{\varepsilon n}{20}.$$

This proves Theorem 1. □

Remark. The constant $\frac{1}{20}$ is obtained by choosing more aggressive parameters in the report's Step 3/6. Further optimization of the same inequalities (still without changing any lemma) can push the constant higher, but $\frac{1}{20}$ is already a factor 12.8 improvement over $\frac{1}{256}$ with a very short argument.