Milking the pp. 31–34 proof for Question 6: a clean constant $c = \frac{3}{40}$

Setting. Let $G = (V, E)$ be an (undirected) graph on $n := |V|$ vertices, with Laplacian L. For $S \subseteq V$, let $G_S = (V, E(S, S))$ be the graph keeping only edges with both endpoints in S, and let L_S be its Laplacian.

Definition 1 (ε-light set)**.** *For $\varepsilon \in (0, 1]$, a set $S \subseteq V$ is called* ε-light *if*

$$\varepsilon L - L_S \succeq 0.$$

This note follows the proof template on pp. 31–34 (Steps 1–6), changing only the numerical choices in Step 3 and the arithmetic in Step 6.

Theorem 1 (Milked constant 3/40)**.** *For every graph $G = (V, E)$, every $\varepsilon \in (0, 1]$, and $n = |V|$, there exists an ε-light set $S \subseteq V$ with*

$$|S| \;\geq\; \frac{3}{40}\,\varepsilon\, n.$$

Proof. We mirror the six steps from pp. 31–34.

Step 1: Normalization. If $E = \emptyset$, then $L = 0$ and $S := V$ is trivially ε-light. Assume henceforth $E \neq \emptyset$. Let $d := \operatorname{rank}(L)$; in particular $1 \leq d \leq n-1$. As in pp. 31–34, work on range(L) and write

$$M_S \;:=\; L^{-1/2} L_S L^{-1/2} \quad \text{(PSD on range}(L)\text{)}.$$

Then $L_S \preceq \varepsilon L$ is equivalent to $M_S \preceq \varepsilon I$ on range(L).

Step 2: Partial coloring and the matrix process. Fix an integer $r \geq 1$ (the number of colors). At time t ($0 \leq t \leq k$) we have colored a set $T \subseteq V$ with $|T| = t$ by a map $\mathrm{col} : T \to \{1, \ldots, r\}$. Define M_t exactly as on p. 33 (equation (37)): it is the sum of contributions from edges whose endpoints are already colored *and* receive the same color. If $R := V \setminus T$ is the set of uncolored vertices and $m := |R| = n - t$, then for $v \in R$ and $\gamma \in \{1, \ldots, r\}$ let B_v^γ be the prospective increment (equation (38)), so that coloring v by γ yields $M_{t+1} = M_t + B_v^\gamma$.

Step 3: Parameter choices (the only change vs. pp. 31–34). Assume $n \geq 3$ for now; the small-n patch is at the end. Set

$$k \;:=\; \left\lceil \frac{3n}{8} \right\rceil, \qquad \delta \;:=\; \frac{\varepsilon}{n}, \qquad u_0 \;:=\; \left(\frac{5}{8} - \frac{1}{n}\right)\varepsilon.$$

Let

$$m_\star \;:=\; n - k + 1, \qquad r \;:=\; \left\lceil \frac{n}{u_0\, m_\star} \;+\; \frac{1}{\delta\, m_\star} \right\rceil.$$

(Thus r depends on n and ε.)

Step 4: Inductive barrier invariant (same argument, new arithmetic). Let $u_t := u_0 + t\delta$ and let $\Phi_u(\cdot)$ be the barrier potential from pp. 31–34. Maintain the invariant (equation (39) on p. 33)

$$M_t \prec u_t I \qquad \text{and} \qquad \Phi_{u_t}(M_t) \le \Phi_{u_0}(0) = \frac{d}{u_0}.$$

The proof on pp. 33–34 shows that if the average in (42) is < 1 then there exists some choice of $(v, \gamma) \in R \times \{1, \dots, r\}$ for which the barrier condition holds, hence the invariant propagates.

So it suffices to bound the average (42). For $t < k$ we have $m = n - t \ge n - k + 1 = m_\star$ and also $d \le n$. Therefore the average in (42) is at most

$$\frac{d/u_0}{mr} + \frac{1}{\delta m r} \;\le\; \frac{n/u_0}{m_\star r} + \frac{1}{\delta m_\star r} \;\le\; \frac{n/u_0}{m_\star\left(\frac{n}{u_0 m_\star} + \frac{1}{\delta m_\star}\right)} + \frac{1}{\delta m_\star\left(\frac{n}{u_0 m_\star} + \frac{1}{\delta m_\star}\right)} \;\le\; 1,$$

where we used the definition of r as the ceiling of the displayed sum. Hence the induction runs for $t = 0, 1, \dots, k$.

Step 5: Extracting an ε-light set (same argument, new u_k). After k steps, the colored set T (with $|T| = k$) is partitioned into color classes $S_1, \dots, S_r$. As on p. 34,

$$M_k \;=\; \sum_{a=1}^{r} L^{-1/2} L_{S_a} L^{-1/2} \quad \text{on range}(L).$$

From the invariant, $M_k \preceq u_k I$ with $u_k = u_0 + k\delta$. Using $k = \lceil 3n/8 \rceil \le 3n/8 + 1$,

$$u_k = \left(\frac{5}{8} - \frac{1}{n}\right)\varepsilon + k \cdot \frac{\varepsilon}{n} \le \left(\frac{5}{8} - \frac{1}{n}\right)\varepsilon + \left(\frac{3}{8} + \frac{1}{n}\right)\varepsilon = \varepsilon.$$

Let S be the largest color class. Since each summand is PSD,

$$L^{-1/2} L_S L^{-1/2} \;\preceq\; M_k \;\preceq\; \varepsilon I \quad \text{on range}(L).$$

Equivalently, $L_S \preceq \varepsilon L$, i.e. S is ε-light.

Step 6: Size lower bound (new arithmetic). Among the k colored vertices, the largest color class has size at least k/r, so

$$|S| \;\ge\; \frac{k}{r}. \tag{1}$$

We now show $k/r \ge \frac{3}{40}\varepsilon n$ for all $n \ge 3$.

Write $m_\star = n - k + 1$ and note that $u_0 = \alpha\varepsilon$ with $\alpha = \frac{5}{8} - \frac{1}{n}$. Then

$$\frac{n}{u_0 m_\star} + \frac{1}{\delta m_\star} = \frac{n}{\varepsilon m_\star}\left(1 + \frac{1}{\alpha}\right) = \frac{n}{\varepsilon m_\star} \cdot \frac{13n - 8}{5n - 8}.$$

Define the n-dependent constant

$$C_n \;:=\; \frac{n}{m_\star} \cdot \frac{13n - 8}{5n - 8} \qquad \text{so that} \qquad r = \left\lceil \frac{C_n}{\varepsilon} \right\rceil.$$

Case 1: $r \le 5$. Then (**??**) gives $|S| \ge k/5$. Since $k = \lceil 3n/8 \rceil \ge 3n/8$ and $\varepsilon \le 1$,

$$|S| \;\ge\; \frac{3n}{40} \;\ge\; \frac{3}{40}\varepsilon n.$$

Case 2: $r \geq 6$. Because $r = \lceil C_n/\varepsilon \rceil \geq 6$, we have $C_n/\varepsilon > 5$, i.e. $\varepsilon < C_n/5$. Using (**??**) and $r \geq 6$,

$$|S| \;\geq\; \frac{k}{r} \;\geq\; \frac{k}{6}.$$

Therefore it suffices to show

$$\frac{k}{6} \;\geq\; \frac{3}{40}\varepsilon n \quad \text{whenever} \quad \varepsilon < \frac{C_n}{5}.$$

Indeed, $\varepsilon < C_n/5$ implies

$$\frac{3}{40}\varepsilon n \;\leq\; \frac{3}{40} \cdot \frac{C_n}{5}\, n \;=\; \frac{3C_n}{200}\, n,$$

so it is enough to prove $\frac{k}{6} \geq \frac{3C_n}{200} n$, i.e.

$$C_n \;\leq\; \frac{100}{9} \cdot \frac{k}{n}. \tag{2}$$

The next lemma establishes (**??**) for all $n \geq 3$, completing Case 2 and thus Step 6.

Lemma 1. *Let $n \geq 3$ and $k = \lceil 3n/8 \rceil$, $m_\star = n - k + 1$. Then (**??**) holds:*

$$\frac{n}{m_\star} \cdot \frac{13n - 8}{5n - 8} \;\leq\; \frac{100}{9} \cdot \frac{k}{n}.$$

Proof. Write $n \equiv r \pmod 8$ with $r \in \{0, 1, \dots, 7\}$. Then one checks

$$k = \left\lceil \frac{3n}{8} \right\rceil = \frac{3n + b_r}{8}, \qquad m_\star = n - k + 1 = \frac{5n + s_r}{8},$$

where

$$(b_0, \dots, b_7) = (0, 5, 2, 7, 4, 1, 6, 3), \qquad (s_0, \dots, s_7) = (8, 3, 6, 1, 4, 7, 2, 5).$$

Substituting into (**??**) and clearing denominators (all positive for $n \geq 3$) yields the cubic inequality

$$P_r(n) \;:=\; 25(3n + b_r)(5n + s_r)(5n - 8) \;-\; 144n^2(13n - 8) \;\geq\; 0.$$

Expanding gives

$$\begin{aligned}
P_0(n) &= 3n^3 + 1152n^2 - 4800n, \\
P_1(n) &= 3n^3 + 2402n^2 - 4925n - 3000, \\
P_2(n) &= 3n^3 + 1652n^2 - 4100n - 2400, \\
P_3(n) &= 3n^3 + 2902n^2 - 6725n - 1400, \\
P_4(n) &= 3n^3 + 2152n^2 - 4400n - 3200, \\
P_5(n) &= 3n^3 + 1402n^2 - 4325n - 1400, \\
P_6(n) &= 3n^3 + 2652n^2 - 5700n - 2400, \\
P_7(n) &= 3n^3 + 1902n^2 - 4125n - 3000.
\end{aligned}$$

For each r, the derivative $P_r'(n) = 9n^2 + 2a_r n + b_r'$ has positive coefficients and in fact $P_r'(n) > 0$ for all $n \geq 3$, so P_r is strictly increasing on $[3, \infty)$. Hence it suffices to check $P_r(n_0) > 0$ at the smallest admissible $n_0 \geq 3$ in the residue class $n \equiv r \pmod 8$:

$$n_0 = \begin{cases} 8, & r = 0, \\ 9, & r = 1, \\ 10, & r = 2, \\ r, & r \in \{3, 4, 5, 6, 7\}. \end{cases}$$

A direct evaluation gives $P_r(n_0) > 0$ in all eight cases, so $P_r(n) \geq 0$ for all admissible n in that residue class. This proves (**??**). □

Combining Cases 1 and 2 proves $|S| \geq \frac{3}{40}\varepsilon n$ for all $n \geq 3$.

Small-n patch (two cases). For $n = 1$ or $n = 2$, pick any vertex v and set $S = \{v\}$. Then $L_S = 0$, so S is ε-light. Moreover, $|S| = 1 \geq \frac{3}{40}\varepsilon n$ since $\varepsilon \leq 1$ and $\frac{3}{40}n \leq \frac{3}{20} < 1$ for $n \leq 2$.

This completes the proof. $\square$