Milking the pp. 31–34 proof for Question 6: an improved constant for $n \geq 6$

Setting. Let $G = (V, E)$ be a (simple, unweighted) graph on $n := |V|$ vertices with Laplacian L. For a subset $S \subseteq V$, let L_S be the Laplacian of the induced subgraph $G[S]$. A set S is called *ε-light* if

$$\varepsilon L - L_S \succeq 0.$$

Goal (Question 6). Find an absolute constant $c > 0$ such that for every graph G and every $\varepsilon \in (0, 1)$ there exists an ε-light set S with $|S| \geq c\varepsilon n$.

This note keeps *Steps 1–5* of the pp. 31–34 proof unchanged (barrier method + partial coloring) and only replaces the numerical choices in *Step 3/6.* The result is a slightly larger constant when one restricts to $n \geq 6$.

Theorem 1 (Milked constant for $n \geq 6$)**.** *There is an absolute constant*

$$c = \frac{151}{2000} = 0.0755$$

such that for every graph G on $n \geq 6$ vertices and every $\varepsilon \in (0, 1)$ there exists an ε-light set $S \subseteq V$ with

$$|S| \ \geq \ c\varepsilon n.$$

Step 1–2 (unchanged)

We use the same barrier potential, barrier update lemma, and partial-coloring framework as on pp. 31–34. In particular, in the notation of that proof, after t colored vertices the number of uncolored vertices is $m = n - t$, and the averaged barrier inequality is bounded by

$$\text{average} \ \leq \ \frac{d/u_0}{mr} + \frac{1}{\delta mr}, \tag{1}$$

where $d \leq n$ is the number of nonzero eigenvalues of L.

Step 3 (new parameter choice)

Fix $n \geq 6$ and $\varepsilon \in (0, 1)$. Set

$$\alpha := \frac{23}{50}, \qquad \beta := \frac{117}{200}, \qquad 1 - \beta = \frac{83}{200}.$$

Choose the number of colored vertices

$$k := \lceil \alpha n \rceil,$$

and set the initial barrier and step size by

$$u_0 := \beta\,\varepsilon, \qquad \delta := \frac{\varepsilon - u_0}{k} = \frac{(1-\beta)\varepsilon}{k}.$$

Then the barrier at time k is exactly

$$u_k = u_0 + k\delta = \varepsilon. \tag{2}$$

Let

$$m_\star := n - k + 1 \quad (\text{so that } m = n - t \geq m_\star \text{ for all } t < k).$$

Define

$$C := \frac{n}{\beta m_\star} + \frac{k}{(1-\beta)m_\star}, \qquad r := \left\lceil \frac{C}{\varepsilon} \right\rceil. \tag{3}$$

Step 4 (feasibility, unchanged argument)

For every $t < k$ we have $m \geq m_\star$ and $d \leq n$. Using (**??**) and the definitions above,

$$\frac{d/u_0}{mr} + \frac{1}{\delta mr} \;\leq\; \frac{n/u_0}{m_\star r} + \frac{1}{\delta m_\star r} = \frac{1}{r}\left(\frac{n}{u_0 m_\star} + \frac{1}{\delta m_\star}\right) = \frac{1}{r}\cdot\frac{C}{\varepsilon} \;\leq\; 1,$$

because $r = \lceil C/\varepsilon \rceil$. Therefore the pp. 31–34 procedure can be iterated for k steps (coloring k vertices with r colors) while maintaining the barrier invariant.

Step 5 (conclusion that a color class is ε-light, unchanged)

At time k, the barrier satisfies (**??**), hence the invariant yields $M_k \preceq \varepsilon I$. As in pp. 31–34, for each color $a \in [r]$ let S_a be the vertex set of that color. Then each S_a is ε-light, i.e. $L_{S_a} \preceq \varepsilon L$.

Step 6 (size bound with $c = 151/2000$)

Let S be a largest color class. Then $|S| \geq k/r$.

We first record a small calculus fact about the ceiling function.

Lemma 1. *Let $C \in (0,6)$ and $\varepsilon \in (0,1]$. Set $r = \lceil C/\varepsilon \rceil$. Then*

$$r\varepsilon \;\leq\; \max\left\{6,\ \frac{7C}{6}\right\}.$$

Proof. For $j \geq 6$, on the interval $\varepsilon \in (C/(j+1),\, C/j]$ we have $r = j+1$ and hence $r\varepsilon = (j+1)\varepsilon$ is maximized at the right endpoint, giving

$$r\varepsilon \leq (j+1)\frac{C}{j} = C\left(1 + \frac{1}{j}\right).$$

This bound decreases with j, so among all $j \geq 6$ it is maximized at $j = 6$, yielding $r\varepsilon \leq 7C/6$. On the remaining interval $\varepsilon \in (C/6, 1]$ we have $r \leq 6$, hence $r\varepsilon \leq 6$. □

We claim that with our choice (**??**) one has $C < 6$ for all $n \geq 6$. Indeed, since $k = \lceil \alpha n \rceil \leq \alpha n + 1$ and $m_\star = n - k + 1 \geq (1 - \alpha)n$,

$$C = \frac{n}{\beta m_\star} + \frac{k}{(1-\beta)m_\star} \leq \frac{1}{(1-\alpha)} \left(\frac{1}{\beta} + \frac{\alpha}{1-\beta} + \frac{1}{(1-\beta)n} \right).$$

With $\alpha = 23/50$, $\beta = 117/200$, $1 - \beta = 83/200$ and $n \geq 6$, the right-hand side equals

$$\frac{50}{27} \left(\frac{200}{117} + \frac{92}{83} + \frac{200}{83n} \right) \leq \frac{50}{27} \left(\frac{200}{117} + \frac{92}{83} + \frac{200}{83 \cdot 6} \right) < 6.$$

Therefore Lemma **??** applies.

Hence

$$\frac{|S|}{\varepsilon n} \;\geq\; \frac{k}{n} \cdot \frac{1}{r\varepsilon} \;\geq\; \frac{k}{n} \cdot \frac{1}{\max\{6,\, 7C/6\}} = \min\left\{ \frac{k}{6n},\, \frac{6k}{7Cn} \right\}.$$

For $n \geq 6$ we have $k = \lceil (23/50)n \rceil \geq 3$, so $k/(6n) \geq 3/(36) = 1/12 > 151/2000$. It remains to show

$$\frac{6k}{7Cn} \;\geq\; \frac{151}{2000}. \tag{4}$$

Write $k = \alpha n + t$ with $t \in [0, 1)$ and note that then $m_\star = n - k + 1 = (1 - \alpha)n + (1 - t)$. Also, with $\beta = 117/200$ and $1 - \beta = 83/200$,

$$C = \frac{n}{\beta m_\star} + \frac{k}{(1-\beta)m_\star} = \frac{1}{m_\star} \left(\frac{200}{117} n + \frac{200}{83} k \right).$$

After substituting this into (**??**) and clearing denominators, (**??**) is equivalent to

$$6(\alpha n + t)\big((1-\alpha)n + (1-t)\big) \;\geq\; \frac{1057}{2000}\, n \left(\frac{200}{117} n + \frac{200}{83}(\alpha n + t) \right). \tag{5}$$

A direct expansion shows that the difference "(left) $-$ (right)" equals

$$\frac{28501}{24277500}\, n^2 - \frac{3293}{4150}\, nt + \frac{69}{25}\, n - 6t^2 + 6t.$$

As a function of t this is a concave quadratic (coefficient of t^2 is -6), hence its minimum over $t \in [0, 1]$ is attained at an endpoint. At $t = 0$ the expression equals $\frac{28501}{24277500} n^2 + \frac{69}{25} n > 0$. At $t = 1$ it equals $\frac{28501}{24277500} n^2 + \left(\frac{69}{25} - \frac{3293}{4150} \right) n > 0$. Therefore the difference is positive for all $n \geq 1$ and all $t \in [0, 1]$, proving (**??**), hence (**??**).

Combining the two lower bounds shows $|S|/(\varepsilon n) \geq 151/2000$, which completes the proof of Theorem **??**. □