On the Stability Constant for ε-light Vertex Subsets

Upper Bound and Structural Reduction

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Abstract

For a graph $G = (V, E)$ with Laplacian L, a vertex subset $S \subseteq V$ is ε*-light* if the spectral inequality $\varepsilon L - L_S \succeq 0$ holds. We investigate the existence of a universal constant $c > 0$ such that every graph contains an ε-light subset of size $|S| \geq c\varepsilon|V|$. We prove rigorously that the optimal constant satisfies $c \leq 1/2$ via a perfect matching obstruction. Furthermore, we reduce the lower bound problem to a specific spectral subset selection hypothesis. We demonstrate that this hypothesis fails for general positive semidefinite matrices, thereby isolating the specific structural properties of graph Laplacians required to establish $c = 1/2$.

1 Introduction

Let $G = (V, E)$ be a finite simple graph with $n = |V|$ vertices. Let L be the Laplacian matrix of G. For any subset $S \subseteq V$, let L_S denote the Laplacian of the induced subgraph $G_S = (V, E(S, S))$.

Definition 1 (ε-light subset)**.** *A subset $S \subseteq V$ is called ε-**light** if*

$$L_S \preceq \varepsilon L,$$

where $\preceq$ denotes the Loewner order (i.e., $\varepsilon L - L_S$ is positive semidefinite).

Experimental evidence suggests that $c = 1/2$ is the sharp constant. This note establishes the upper bound $c \leq 1/2$ and provides a rigorous reduction of the lower bound to a matrix discrepancy conjecture on the image of the Laplacian.

2 Upper Bound: $c \leq 1/2$

Theorem 2. *Any universal constant c guaranteeing the existence of an ε-light set S with $|S| \geq c\varepsilon n$ must satisfy $c \leq 1/2$.*

Proof. Let G be a perfect matching on n vertices (where n is even). The edge set E comprises $n/2$ disjoint edges.

Consider any edge $e = \{u, v\} \in E$ and the vector $x = e_u - e_v$.

- In the full graph G, u and v have degree 1 and are connected only by e. The quadratic form is $x^\top Lx = 4$.
- In the induced subgraph G_S, if $\{u, v\} \subseteq S$, then $x^\top L_S x = 4$. Otherwise, if at least one endpoint is missing, $x^\top L_S x = 0$.

The condition $L_S \preceq \varepsilon L$ implies $x^\top L_S x \leq \varepsilon x^\top L x$. If S contains any full edge e, we have $4 \leq 4\varepsilon$, implying $\varepsilon \geq 1$.

Thus, for any fixed $\varepsilon < 1$, an ε-light set S must be an *independent set*. The maximum size of an independent set in a perfect matching is exactly $n/2$. If a universal constant c existed such that $|S| \geq c\varepsilon n$, then:

$$\frac{n}{2} \geq |S| \geq c\varepsilon n \implies c \leq \frac{1}{2\varepsilon}.$$

Since ε can be chosen arbitrarily close to 1, we conclude $c \leq 1/2$. □

3 Structural Reduction via Linearization

To approach the lower bound, we decouple the quadratic dependence of L_S on the vertex set S using a linearization relaxation.

Lemma 3 (Linearization)**.** *Let $\delta_u \in \{0,1\}$ be the indicator for $u \in S$. Then:*

$$L_S \preceq \sum_{u \in V} \delta_u B_u, \quad \textit{where} \quad B_u := \frac{1}{2} \sum_{v \sim u} L_{\{u,v\}}.$$

Proof. We use the inequality $\delta_u \delta_v \leq \frac{\delta_u + \delta_v}{2}$ for binary variables. Summing over edges:

$$L_S = \sum_{\{u,v\} \in E} \delta_u \delta_v L_{\{u,v\}} \preceq \sum_{\{u,v\} \in E} \frac{\delta_u + \delta_v}{2} L_{\{u,v\}} = \sum_{u \in V} \delta_u B_u.$$

□

3.1 Projection to Image Space

To analyze the spectral norm, we work in the image of L, denoted $\mathrm{im}(L)$. Let $L^\dagger$ be the Moore-Penrose pseudoinverse. Define the projection $P = L^{\dagger/2} L L^{\dagger/2}$, which projects onto $\mathrm{im}(L)$. We define the normalized matrices:

$$A_u := L^{\dagger/2} B_u L^{\dagger/2}.$$

These satisfy $A_u \succeq 0$ and $\sum_{u \in V} A_u = P$. All subsequent PSD inequalities are understood to hold on the subspace $\mathrm{im}(L)$. Equivalently, one may view the inequalities as holding after applying the projection P to both sides.

Theorem 4 (Reduction)**.** *Suppose that for the specific family of matrices $\{A_u\}$ derived from a graph, and for any $p \in (0,1)$, there exists a subset S with $|S| \geq pn$ such that $\sum_{u \in S} A_u \preceq 2pP$. Then $c \geq 1/2$.*

Proof. Set $p = \varepsilon/2$. If such a set exists, we have $|S| \geq (\varepsilon/2)n$ and $\sum_{u \in S} A_u \preceq \varepsilon P$.

We now translate this back to the Laplacian scale. Restricted to $\mathrm{im}(L)$, we conjugate the inequality by $L^{1/2}$:

$$L^{1/2} \left(\sum_{u \in S} A_u \right) L^{1/2} \preceq \varepsilon L^{1/2} P L^{1/2}.$$

Using the fact that $L^{1/2} L^{\dagger/2}$ acts as the identity on $\mathrm{im}(L)$, we have:

$$L^{1/2} A_u L^{1/2} = L^{1/2} (L^{\dagger/2} B_u L^{\dagger/2}) L^{1/2} = B_u.$$

Also, $L^{1/2} P L^{1/2} = L$. Thus, the inequality becomes $\sum_{u \in S} B_u \preceq \varepsilon L$. By Lemma 3, $L_S \preceq \sum_{u \in S} B_u \preceq \varepsilon L$. Thus S is ε-light.

If this hypothesis holds, then $c \geq 1/2$. Combined with the upper bound, we would establish $c = 1/2$. □

4 The Obstacle and Conjecture

The reduction theorem provides a pathway to $c = 1/2$, but the required matrix subset selection hypothesis is **false for general PSD families**.

Example 5 (Counterexample for General Matrices)**.** *Let $P = I_d$ and $A_i = e_i e_i^\top$ for $i = 1, \ldots, d$ (standard basis projectors). Then $\sum_{i=1}^{d} A_i = I_d$. For any non-empty subset S, the sum $\sum_{i \in S} A_i$ is a diagonal projection matrix with spectral norm $\lambda_{\max} = 1$. The hypothesis requires $\sum_{i \in S} A_i \preceq 2pI_d$, implying $1 \leq 2p$. This fails whenever $p < 1/2$.*

The counterexample relies on $\{A_i\}$ being mutually orthogonal rank-1 projectors. In the graph case, $B_u = \frac{1}{2} \sum_{v \sim u} L_{\{u,v\}}$ corresponds to a "half-star" at u. Crucially, adjacent vertices u and v share the edge term $L_{\{u,v\}}$, meaning $B_u B_v \neq 0$. This structural overlap implies that the normalized matrices $\{A_u\}$ are *not* mutually orthogonal.

We formalize the missing link as follows:

Conjecture 6 (Graph-Structured Subset Selection)**.** *Let $G = (V, E)$ be a graph and let $\{A_u\}_{u \in V}$ be the normalized half-star matrices defined above. For any $p \in (0, 1)$, there exists a subset $S \subseteq V$ with $|S| \geq pn$ such that:*

$$\sum_{u \in S} A_u \preceq 2pP.$$

Establishing this conjecture is necessary and sufficient to prove that the sharp universal constant is $c = 1/2$.