Problem 07: Improved Proof (Smith Theory)

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Problem

Let Γ be a uniform lattice in a real semisimple Lie group and assume that Γ contains an element of order 2. Can Γ be the fundamental group of a closed (compact, boundaryless) manifold M whose universal cover $\widetilde{M}$ is acyclic?

Answer: No. We answer the question in the negative by proving a stronger topological result: if a closed manifold M has an integrally acyclic universal cover, then $\pi_1(M)$ is torsion-free. Equivalently, no nontrivial finite cyclic group can occur as a subgroup of $\pi_1(M)$. The "lattice" and "semisimple" hypotheses are irrelevant to this topological obstruction.

Proof

We proceed in four steps:

1. Show that integral acyclicity implies mod-p acyclicity.
2. Show that deck transformations must act freely.
3. Establish the isomorphism between the fundamental group and the deck group.
4. Show via Smith Theory that finite cyclic actions on mod-p acyclic spaces cannot be free.

Lemma 1 (Integral acyclicity implies mod-p acyclicity)**.** *If a space X is integrally acyclic (i.e., $\widetilde{H}_i(X;\mathbb{Z}) = 0$ for all i), then for every prime p, it is mod-p acyclic ($\widetilde{H}_i(X;\mathbb{Z}/p) = 0$ for all i).*

Proof. For $i > 0$, the Universal Coefficient Theorem gives a split exact sequence:

$$0 \to H_i(X;\mathbb{Z}) \otimes \mathbb{Z}/p \to H_i(X;\mathbb{Z}/p) \to \operatorname{Tor}(H_{i-1}(X;\mathbb{Z}), \mathbb{Z}/p) \to 0.$$

If X is integrally acyclic, then $H_i(X;\mathbb{Z}) = 0$ for all $i > 0$ and $H_0(X;\mathbb{Z}) \cong \mathbb{Z}$. Thus, the tensor and Tor terms vanish for all $i > 0$, implying $H_i(X;\mathbb{Z}/p) = 0$ for $i > 0$. For $i = 0$, $H_0(X;\mathbb{Z}/p) \cong H_0(X;\mathbb{Z}) \otimes \mathbb{Z}/p \cong \mathbb{Z} \otimes \mathbb{Z}/p \cong \mathbb{Z}/p$. Thus, the reduced homology $\widetilde{H}_0(X;\mathbb{Z}/p) = 0$. □

Lemma 2 (Deck transformations act freely)**.** *Let $\pi : \widetilde{M} \to M$ be a covering map with $\widetilde{M}$ connected, and let $f : \widetilde{M} \to \widetilde{M}$ be a deck transformation. If f has a fixed point, then $f = \mathrm{id}_{\widetilde{M}}$.*

Proof. Assume $f(x) = x$. Let $y \in \widetilde{M}$ be arbitrary. Since $\widetilde{M}$ is connected (and locally path-connected as a manifold), it is path-connected. Choose a path $\alpha : [0,1] \to \widetilde{M}$ with $\alpha(0) = x$ and $\alpha(1) = y$. Then $\pi \circ \alpha$ is a path in M starting at $\pi(x)$. Because f is a deck transformation, we have $\pi \circ f = \pi$. Thus:

$$\pi \circ (f \circ \alpha) = (\pi \circ f) \circ \alpha = \pi \circ \alpha.$$

This means α and $f \circ \alpha$ are both lifts of the same path $\pi \circ \alpha$ starting at the same point x (since $f(\alpha(0)) = f(x) = x$). By the uniqueness of path lifting, $f \circ \alpha = \alpha$. Evaluating at $t = 1$ gives $f(y) = y$. Since y was arbitrary, $f = \mathrm{id}_{\widetilde{M}}$. □

Lemma 3 (Deck group of the universal cover)**.** *Let $\pi : \widetilde{M} \to M$ be the universal covering of a connected manifold M and fix a basepoint $\tilde{x} \in \widetilde{M}$ with $x = \pi(\tilde{x})$. Then the map $\Phi : \pi_1(M, x) \to \mathrm{Deck}(\widetilde{M}/M)$ defined by sending a loop class $[\ell]$ to the unique deck transformation taking $\tilde{x}$ to the endpoint of the lift of ℓ starting at $\tilde{x}$ is a group isomorphism. In particular, γ has order p if and only if the corresponding deck transformation has order p.*

Proof. This is standard covering-space theory. The map Φ is well-defined by the uniqueness of path lifting and is a homomorphism by the concatenation of loops. Injectivity and surjectivity follow because $\widetilde{M}$ is universal (simply connected) and the deck group acts simply transitively on each fiber. (See, e.g., Hatcher, *Algebraic Topology*, Section 1.3). □

Lemma 4 (Smith Fixed Point Theorem (Manifold Case))**.** *Let p be a prime and let X be a finite-dimensional topological manifold. Assume $\widetilde{H}_i(X; \mathbb{Z}/p) = 0$ for all i. If the cyclic group C_p acts on X by homeomorphisms, then the fixed point set X^{C_p} is nonempty.*

Reference: Bredon, *Introduction to Compact Transformation Groups*, Ch. III, or tom Dieck, *Transformation Groups*.

Theorem 5. *Let M be a closed manifold. Let p be a prime. If the universal cover $\widetilde{M}$ is mod-p acyclic, then $\pi_1(M)$ contains no element of order p.*

Proof. Suppose $\pi_1(M)$ contains an element γ of order p. Let $f : \widetilde{M} \to \widetilde{M}$ be the associated deck transformation. By Lemma 3, the isomorphism $\pi_1(M) \cong \mathrm{Deck}(\widetilde{M}/M)$ implies that since $\gamma \neq e$, we have $f \neq \mathrm{id}$ and the order of f is p.

By Lemma 2, a deck transformation with a fixed point must be the identity. Thus, $f \neq \mathrm{id}$ implies $\mathrm{Fix}(f) = \emptyset$.

On the other hand, since f has order p, it generates an action of the cyclic group $C_p = \langle f \rangle$ on $\widetilde{M}$ by homeomorphisms. Since $\widetilde{M}$ is mod-p acyclic, Lemma 4 implies $\mathrm{Fix}(f) \neq \emptyset$.

This is a contradiction. Hence $\pi_1(M)$ contains no element of order p. □

Corollary 6 (No torsion at all under integral acyclicity)**.** *If $\widetilde{M}$ is acyclic over $\mathbb{Z}$, then $\pi_1(M)$ is torsion-free.*

Proof. Suppose $\pi_1(M)$ contains a non-trivial torsion element γ of order $m > 1$. Choose a prime p dividing m. Then $\delta = \gamma^{m/p}$ is an element of order p. Since $\widetilde{M}$ is integrally acyclic, Lemma 1 implies it is mod-p acyclic. By Theorem 5, $\pi_1(M)$ cannot contain an element of order p, contradicting the existence of δ. □

Corollary 7. *If $\widetilde{M}$ is acyclic (over $\mathbb{Z}$) and $\pi_1(M)$ contains an element of order 2, then no such closed manifold M exists.*

Proof. This is a special case of Corollary 6 where $p = 2$. □

Conclusion for the problem

If Γ contains an element of order 2 and $\Gamma \cong \pi_1(M)$ for some closed manifold M, then Corollary 7 shows that $\widetilde{M}$ cannot be acyclic. Hence the answer to the problem is *no*.