Problem 07: Final Version (with Corrections)

Problem

Let Γ be a uniform lattice in a real semisimple Lie group and assume that Γ contains an element of order 2. Can Γ be the fundamental group of a closed (compact, boundaryless) manifold M whose universal cover $\widetilde{M}$ is acyclic over $\mathbb{Q}$, i.e.

$$H_i(\widetilde{M};\mathbb{Q}) = 0 \qquad (i > 0)\,?$$

Answer: No. In fact, the following stronger statement holds, without any "lattice" assumption.

Theorem 1. *Let M be a closed n-manifold with universal cover $\widetilde{M}$. Assume that $H_i(\widetilde{M};\mathbb{Q}) = 0$ for all $i > 0$. Then $\pi_1(M)$ is torsion-free (in particular, it contains no element of order 2).*

Proof. Assume for contradiction that $\pi_1(M)$ contains a non-trivial element γ of finite order. Then $\pi_1(M)$ acts freely and properly discontinuously on $\widetilde{M}$ by deck transformations. Hence the deck transformation

$$f := \gamma : \widetilde{M} \longrightarrow \widetilde{M}$$

is a diffeomorphism without fixed points:

$$\mathrm{Fix}(f) = \emptyset.$$

Since $\widetilde{M}$ is a manifold, it is locally compact and Hausdorff; any homeomorphism (in particular f) is therefore *proper*.

(1) Compactly supported cohomology. Because $\widetilde{M}$ is simply connected, it is orientable. For orientable, noncompact n-manifolds, Poincaré duality with compact supports holds (over $\mathbb{Q}$):

$$H^k_c(\widetilde{M};\mathbb{Q}) \;\cong\; H_{n-k}(\widetilde{M};\mathbb{Q}).$$

From $\mathbb{Q}$-acyclicity we get $H_0(\widetilde{M};\mathbb{Q}) \cong \mathbb{Q}$ and $H_i(\widetilde{M};\mathbb{Q}) = 0$ for $i > 0$. Therefore

$$H^k_c(\widetilde{M};\mathbb{Q}) \cong \begin{cases} \mathbb{Q}, & k = n, \\ 0, & k \neq n. \end{cases}$$

(2) Compactly supported Lefschetz number. For a proper continuous map $g : X \to X$ one defines the compactly supported Lefschetz number

$$L_c(g) := \sum_{k=0}^{n} (-1)^k \operatorname{tr}\big(g^* : H^k_c(X;\mathbb{Q}) \to H^k_c(X;\mathbb{Q})\big).$$

Here $X = \widetilde{M}$, and since $H^k_c = 0$ for $k \neq n$ we obtain

$$L_c(f) = (-1)^n \operatorname{tr}\Big(f^*|_{H^n_c(\widetilde{M};\mathbb{Q})}\Big)\,.$$

Because $H^n_c(\widetilde{M};\mathbb{Q}) \cong \mathbb{Q}$ is one-dimensional, the map f^* acts on H^n_c by multiplication by a scalar $\lambda \in \mathbb{Q}$. Since f has finite order, λ must be a root of unity. The only roots of unity in $\mathbb{Q}$ are ± 1, so $\lambda \in \{1, -1\}$. Hence

$$L_c(f) = \pm 1 \neq 0.$$

(3) Proper Lefschetz–Hopf. A standard form of the Lefschetz–Hopf fixed point theorem for noncompact manifolds (expressed via compactly supported cohomology) states: if $g : X \to X$ is proper and $\mathrm{Fix}(g)$ is compact, then $L_c(g)$ equals the sum of fixed point indices; in particular, if $\mathrm{Fix}(g) = \emptyset$ then $L_c(g) = 0$.

Here $\mathrm{Fix}(f) = \emptyset$ (hence compact), so we must have $L_c(f) = 0$. This contradicts $L_c(f) = \pm 1$. Therefore no non-trivial element of finite order can exist in $\pi_1(M)$.

Hence $\pi_1(M)$ is torsion-free. □

Consequence for Problem 07

If Γ is a uniform lattice (or more generally any group) with 2-torsion, then Γ cannot occur as the fundamental group of a closed manifold whose universal cover is $\mathbb{Q}$-acyclic.

Remarks on the development / corrections

Remark 2 (Why the classical Lefschetz number is not sufficient)**.** *On noncompact spaces, the "classical" Lefschetz number (computed from ordinary homology) is not appropriate. Example: the translation* $t : \mathbb{R} \to \mathbb{R}$, $t(x) = x + 1$, *has no fixed point, while* $\mathbb{R}$ *is* $\mathbb{Q}$*-acyclic and a naive computation would give Lefschetz number* 1*. In contrast, the* compactly supported *Lefschetz number satisfies* $L_c(t) = 0$*, which matches the correct fixed-point behavior.*