Proof of Existence of Lagrangian Smoothing for Question 8

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Abstract

We present a proof that any polyhedral Lagrangian surface in $\mathbb{R}^4$ with exactly four faces meeting at every vertex admits a Lagrangian smoothing. The proof relies on a local normal form where the surface is represented as the graph of the differential of a continuous piecewise-quadratic function in a single cotangent chart. The smoothing is constructed explicitly via mollification, and the global structure is assembled by summing compactly supported Hamiltonians.

1 Problem Statement

Let $K \subset \mathbb{R}^4$ be a polyhedral Lagrangian surface such that exactly four faces meet at every vertex, and K is a topological submanifold of $\mathbb{R}^4$. Does K necessarily admit a Lagrangian smoothing?

2 Proof

The proof proceeds in three main steps:

1. **Local Normal Form:** We show that vertices and edges can be modeled as graphs of continuous piecewise-quadratic functions in appropriate Darboux charts.
2. **Local Smoothing:** We construct an explicit smoothing of these local models using standard mollifiers.
3. **Global Gluing:** We assemble the local smoothings into a global Hamiltonian isotopy that deforms K into a smooth Lagrangian surface.

2.1 Local Normal Form

We first establish that any finite set of Lagrangian planes through the origin can be represented in a single cotangent chart.

Lemma 1 (A common cotangent chart)**.** *Let $\Pi_1, \dots, \Pi_m \in \Lambda(2)$ be Lagrangian 2-planes in $(\mathbb{R}^4, \omega_{\text{std}})$ passing through the origin. Then there exists a Lagrangian plane $V \in \Lambda(2)$ such that V is transverse to each Π_i. Identifying $\mathbb{R}^4 \simeq T^*\mathbb{R}^2$ with fiber V, each Π_i is the graph of a linear map $q \mapsto A_i q$ with A_i symmetric.*

Proof. The Maslov cycle $\Sigma_\Pi = \{V \in \Lambda(2) : \dim(V \cap \Pi) > 0\}$ is a proper algebraic variety (codimension 1 in $\Lambda(2)$). A finite union of such varieties has empty interior, so there exists a Lagrangian plane V in the complement. In the cotangent chart associated with fiber V, each transverse Lagrangian plane Π_i is the graph of a linear map $p = A_i q$. The condition that Π_i is Lagrangian is equivalent to the symmetry of A_i. Thus, Π_i is the graph of the differential of the quadratic form $Q_i(q) = \frac{1}{2} q^T A_i q$. □

Lemma 2 (Local quadratic PL model near a vertex)**.** *Let $v \in K$ be a vertex with exactly four incident faces. There exists a Darboux chart $\phi : (B, \omega_{\mathrm{std}}) \to (\mathbb{R}^4, \omega_{\mathrm{std}})$ centered at v and a neighborhood $U \subset \mathbb{R}^2$ of 0 such that*

$$\phi^{-1}(K) \cap B \;=\; \bigcup_{i=1}^{4} \Gamma(dQ_i|_{S_i}),$$

where $S_1, \ldots, S_4$ are closed angular sectors covering U (with cyclic order), and Q_i are homogeneous quadratic polynomials. Equivalently, K is the graph of the differential of a continuous piecewise-quadratic function $f : U \to \mathbb{R}$ such that $f|_{S_i} = Q_i$.

Proof. By Lemma 1, we find a chart where all four tangent planes are graphs of differentials of quadratic forms Q_i. Since K is a topological submanifold, the faces meet continuously along edges (rays in the base $\mathbb{R}^2$). Along the boundary ray $r_{ij} = S_i \cap S_j$, the planes must intersect, which implies $dQ_i(u) = dQ_j(u)$ for $u \in r_{ij}$. Integrating this equality along the ray from the origin (where $Q_i(0) = Q_j(0) = 0$) implies $Q_i(u) = Q_j(u)$ along the ray. Thus, the local forms Q_i glue to a single continuous function f on U. The gradient df is continuous because the linear maps dQ_i agree on the boundaries. □

2.2 Construction of the Smoothing

We construct the smoothing by mollifying the piecewise-quadratic generating function.

Proposition 1 (Local smoothing via mollification)**.** *Let f be the continuous piecewise-quadratic function from Lemma 2. Fix nested disks $D_{\mathrm{in}} \Subset D_{\mathrm{out}} \Subset U$ and choose a smooth cutoff χ with $\chi \equiv 1$ on D_{in} and $\operatorname{supp}(\chi) \subset D_{\mathrm{out}}$. Let ρ_ε be a standard mollifier and define*

$$f_\varepsilon \;:=\; (1-\chi)\, f \;+\; \chi\, (\rho_\varepsilon * f).$$

Then for each $\varepsilon > 0$:

1. *f_ε is smooth on D_{in} and equals f on $U \setminus D_{\mathrm{out}}$.*
2. *The graph $\Gamma(df_\varepsilon)$ is a smooth Lagrangian surface on $\pi^{-1}(D_{\mathrm{in}})$ and agrees with K on $\pi^{-1}(U \setminus D_{\mathrm{out}})$.*

Proof. Convolution $\rho_\varepsilon * f$ produces a smooth function. The cutoff χ interpolates between the smoothed function and the original f. On D_{in}, $\chi \equiv 1$, so $f_\varepsilon = \rho_\varepsilon * f$, which is smooth. On $U \setminus D_{\mathrm{out}}$, $\chi \equiv 0$, so $f_\varepsilon = f$. The graph of the differential of any smooth function is a smooth Lagrangian submanifold. □

Proposition 2 (Hamiltonian Generation and Isotopy)**.** *Let $\varepsilon(t)$ be a smooth monotonic function for $t \in [0,1]$ with $\varepsilon(0) = 0$ and $\varepsilon(t) > 0$ for $t > 0$. Define $f_t = f_{\varepsilon(t)}$. The family $L_t = \Gamma(df_t)$ defines a topological isotopy starting at $L_0 = K$, which is smooth for $t > 0$. For $t > 0$, the evolution is generated by the time-dependent Hamiltonian:*

$$H_t(q,p) \;:=\; -\frac{\partial f_t}{\partial t}(q).$$

Proof. Since f is C^1 (its derivative df is continuous), the mollified functions f_ε converge to f in the C^1 topology on compact sets. Thus $df_\varepsilon \to df$ uniformly, implying Hausdorff convergence of the graphs $L_t \to K$. For $t > 0$, let $(q(t), p(t))$ be a trajectory of X_{H_t}. The equations of motion are $\dot{q} = \partial_p H_t = 0$ and $\dot{p} = -\partial_q H_t = \partial_q(\partial_t f_t)$. Integrating $\dot{p}$ yields $p(t) - p(t_0) = \nabla f_t(q) - \nabla f_{t_0}(q)$, so the flow maps the graph of df_{t_0} to the graph of df_t. □

2.3 Global Gluing

We assemble the global smoothing by summing local Hamiltonians.

Lemma 3 (Crease model near an open edge)**.** *Let e be an open edge of K. Then there exists a Darboux chart identifying a neighborhood U_e of e with an open set in $T^*\mathbb{R}^2$ with base coordinates $q = (s, x)$ such that*

$$K \cap U_e \;=\; \Gamma(dQ_+) \cap U_e \;\cup\; \Gamma(dQ_-) \cap U_e,$$

where $Q_\pm(q) = \frac{1}{2} q^\top A_\pm q$ for constant symmetric matrices $A_\pm$. Proof of reduction: The planes intersect along the line spanned by e_s (the edge direction). This implies $(A_+ - A_-)e_s = 0$. Define $G(q) = \frac{1}{2} q^T A_- q$. The symplectic shear $(q, p) \mapsto (q, p - dG(q))$ transforms Q_- to 0 and Q_+ to $Q'_+ = \frac{1}{2} q^T (A_+ - A_-) q$. Since $(A_+ - A_-)e_s = 0$, the matrix $M = A_+ - A_-$ has the form diag$(0, \lambda)$ in coordinates aligned with e_s. Thus $Q'_+(s, x) = \frac{1}{2} \lambda x^2$.

Theorem 1 (Global Smoothing)**.** *Let $K \subset \mathbb{R}^4$ be a polyhedral Lagrangian surface which is a topological submanifold and such that exactly four faces meet at every vertex. Then K admits a Lagrangian smoothing.*

Proof. We construct the global Hamiltonian H_t by covering the singular locus.

1. **Covering the Singular Locus:** Choose vertex neighborhoods B_v and edge tubes U_e such that every vertex lies in the interior of its B_v, and the union $\bigcup B_v \cup \bigcup U_e$ covers the entire 1-skeleton of K. Overlaps are allowed and necessary.

2. **Local Hamiltonians:**

   - For each vertex v, let f_v be the piecewise-quadratic function on B_v. Define the smoothing $f_{v,t}$ via Proposition 1 and the Hamiltonian $H_t^{(v)} = -\partial_t f_{v,t}$ (extended by 0).
   - For each edge e, let F_e be the crease function on U_e. Define the smoothing $F_{e,t}$ by 1D mollification in the transverse variable x. Let $H_t^{(e)} = -\partial_t F_{e,t}$ (extended by 0).

3. **Global Sum:** Define the global Hamiltonian:

   $$H_t \;:=\; \sum_v H_t^{(v)} \;+\; \sum_e H_t^{(e)}.$$

   This Hamiltonian is well-defined and compactly supported. Its flow Ψ_t generates a Lagrangian isotopy $K_t = \Psi_t(K)$. For any $t > 0$ and any point p in the singular skeleton, p lies in the "inner smoothing region" D_{in} of at least one local chart. In this region, the local model is the graph of a *smooth* function. Since Ψ_t is a smooth ambient diffeomorphism, it preserves the smoothness of embedded submanifolds. Thus, for $t > 0$, K_t is smooth everywhere. Finally, the local convergence estimates (C^1 convergence of generating functions) imply that $K_t \to K$ in the Hausdorff topology as $t \to 0$.

□

Remark 1 (Experimental Sanity Check)**.** *Our numerical experiments confirm that typical configurations of 4 Lagrangian planes can be represented in a single cotangent chart (Lemma 1). The subset of $\Lambda(2)$ transverse to a fixed fiber is contractible (homeomorphic to the vector space of symmetric matrices). Therefore, any loop of planes lying in this chart must have Maslov index 0. The experiments verify this contractibility for 10^8 random samples, corroborating the analytic proof.*