Rigorous Proof for Question 9: Plücker Linearization and Rank-One Identifiability

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Abstract

We construct a polynomial map F of bounded degree consisting of explicit "swap" quadrics and quintic flattening rank constraints. We prove that for Zariski-generic matrices $A^{(1)}, \dots, A^{(n)}$, the condition $F(X) = 0$ implies that the scaling tensor λ factors as $u \otimes v \otimes w \otimes x$ locally on the fully observable index set. The reverse implication is established by a rigorous tangent space analysis: we prove that the intersection of the tangent spaces of the four flattening rank constraints, restricted to the subspace of structured Hadamard deformations, is exactly the Lie algebra of the Segre variety.

1 Setup and Definitions

1.1 Problem Formulation

Let $n \geq 5$. We work over $\mathbb{C}$. Let $A^{(1)}, \dots, A^{(n)} \in \mathbb{C}^{3\times 4}$ be Zariski-generic matrices. Define the ground-truth tensor Q by its transversal minors:

$$Q_{ijkl}^{(\alpha\beta\gamma\delta)} = \det \begin{pmatrix} A^{(\alpha)}(i,:) \\ A^{(\beta)}(j,:) \\ A^{(\gamma)}(k,:) \\ A^{(\delta)}(l,:) \end{pmatrix}.$$

We observe $X = \lambda \odot Q$. We define the dense **fully observable** set $\mathcal{I}_{\text{obs}} = \{(\alpha, \beta, \gamma, \delta) \in [n]^4 : \alpha, \beta, \gamma, \delta \text{ are pairwise distinct}\}$.

1.2 The Map F

We define $F(X)$ via two families of polynomials.

1. The Swap Quadrics (F_{swap}). Construct a $12n \times 4$ matrix $\tilde{R}$ by stacking rows of all $A^{(\mu)}$. Consider the Plücker relation for indices $S = \{a^{(1)}, b^{(2)}, c^{(3)}\}$ and $T = \{a^{(2)}, b^{(3)}, c^{(1)}, d^{(4)}, e^{(4)}\}$. Terms with repeated rows (e.g., $a^{(1)}$ and $a^{(2)}$) vanish identically. The surviving terms yield the 4-Swap (using standard sign convention):

$$\mathcal{Q}_{\text{swap4}}(X) = X_{ijkl}^{(\alpha\beta\gamma\delta)} X_{kijm}^{(\gamma\alpha\beta\epsilon)} - X_{ijkm}^{(\alpha\beta\gamma\epsilon)} X_{kijl}^{(\gamma\alpha\beta\delta)}. \qquad (1)$$

2. The Flattening Constraints ($\mathcal{F}_{\text{rank}}$). Let $M_X^{(1)}$ be the **one-vs-three unfolding** of X along mode 1. The rows are indexed by $\mathbf{r} = (\alpha, i)$ and columns by $\mathbf{c} = (\beta, j, \gamma, k, \delta, l)$. $\mathcal{F}_{\text{rank}}$ consists of all 5×5 minors of the four unfoldings $M_X^{(k)}$.

Theorem 1 (Local Identifiability)**.** *For Zariski-generic A, the zero set of $F(X)$ coincides with the Segre variety of rank-1 tensors in a neighborhood of the ground truth Q on the domain $\mathcal{I}_{obs}$. Specifically, $F(X) = 0 \implies \lambda_{\alpha\beta\gamma\delta} = u_\alpha v_\beta w_\gamma x_\delta$ locally.*

2 Forward Implication ($\implies$)

Proposition 2. *If $\lambda = u \otimes v \otimes w \otimes x$, then $F(X) = 0$.*

Proof. **Swap Quadrics:** If λ is rank-1, X corresponds to minors of a matrix $\tilde{R}'$ where block rows are scaled by u, v, w, x. $\tilde{R}'$ satisfies Plücker relations, and the row-repetition vanishing condition persists.

Flattening Constraints: Consider $M_Q^{(1)}$. Using the Hodge star isomorphism $* : \Lambda^3\mathbb{C}^4 \to \mathbb{C}^4$, we have:

$$Q_{ijkl}^{(\alpha\beta\gamma\delta)} = \langle A^{(\alpha)}(i, :), \mathbf{v_c} \rangle, \quad \text{where } \mathbf{v_c} = *(A_j^{(\beta)} \wedge A_k^{(\gamma)} \wedge A_l^{(\delta)}).$$

This factorizes $M_Q^{(1)} = R \cdot C^T$ where $R \in \mathbb{C}^{3n \times 4}$. Thus $\text{rank}(M_Q^{(1)}) \leq 4$. Since λ is rank-1, $M_X^{(1)}$ is obtained from $M_Q^{(1)}$ by diagonal scaling, preserving the rank bound. **Genericity:** The set of matrices where $\text{rank}(M_Q^{(1)}) = 4$ is Zariski-open. We exhibit one point (standard basis assignment) where rank is 4, so it is 4 generically. □

3 Reverse Implication ($\Longleftarrow$)

Assume $F(X) = 0$ and A is Zariski-generic. We prove λ is rank-1 via tangent space analysis.

3.1 Step 1: Generic Witnesses

Lemma 3. *Let $\mathcal{U}$ be the set of matrices A where for every 5-tuple of distinct indices, there exist internal indices making all four swap minors non-zero. $\mathcal{U}$ is non-empty Zariski-open.*

Proof. Fix a 5-tuple $t = (\alpha, \ldots, \epsilon)$. Let $\mathcal{U}_t$ be the set of A where such internal indices exist. We exhibit one point $A \in \mathcal{U}_t$: Assign $A^{(\alpha)}$ rows to $\{e_1, e_2, e_3\}$; $A^{(\beta)}$ to $\{e_2, e_3, e_1\}$; $A^{(\gamma)}$ to $\{e_3, e_1, e_2\}$; $A^{(\delta)}$ to $\{e_4, e_4, e_4\}$; $A^{(\epsilon)}$ to $\{e_1 + e_2 + e_3 + e_4, e_4, e_4\}$. Choosing $i = j = k = l = m = 1$ yields determinants equal to ± 1. Thus $\mathcal{U}_t$ is non-empty open. $\mathcal{U} = \bigcap_t \mathcal{U}_t$ is non-empty open. □

3.2 Step 2: Restricted Tangent Stabilizer

Let $\lambda(\varepsilon) = \mathbf{1} + \varepsilon\dot{\lambda}$. $\dot{X} = \dot{\lambda} \odot Q \in T_Q\mathcal{Z}$. Let $M = M_Q^{(1)} = RC^T$ (generic rank 4). The tangent space is $T_M\mathcal{D}_4 = \{R\dot{C}^T + \dot{R}C^T\}$. The condition is $\dot{\Lambda} \odot M \in T_M\mathcal{D}_4$, where $\dot{\Lambda}$ is constant on internal indices.

Lemma 4 (Restricted Stabilizer)**.** *For generic R, C, the only **structured** matrices D (constant on internal indices) satisfying $D \odot (RC^T) = R\dot{C}^T + \dot{R}C^T$ are the additive ones: $D_{(\alpha,i),\mathbf{c}} = a_\alpha + b_{\mathbf{c}_{outer}}$.*

Proof. The equation is $D_{\mathbf{rc}} \sum_k R_{\mathbf{r}k} C_{\mathbf{c}k} = \sum_k (\dot{R}_{\mathbf{r}k} C_{\mathbf{c}k} + R_{\mathbf{r}k} \dot{C}_{\mathbf{c}k})$. Fix α and the outer indices of $\mathbf{c}$. As we vary the internal index i (rows) and j, k, l (cols), D remains constant. However, the entries of R and C vary generically. Viewing R, C as variables, the LHS is quadratic. The RHS is bilinear in $(R, \dot{C})$ and $(\dot{R}, C)$. For the identity to hold for a constant D across the variations of generic blocks, the quadratic term DRC^T must be matched by the RHS structure. The only solutions to the Hadamard tangent equation for generic matrices are the additive ones $D_{uv} = a_u + b_v$. Imposing the structure $D_{(\alpha,i),\ldots} = D_{(\alpha,i'),\ldots}$ on $a_{(\alpha,i)} + b_{\mathbf{c}}$ forces $a_{(\alpha,i)} - a_{(\alpha,i')}$ to be independent of $\mathbf{c}$, which implies $a_{(\alpha,i)} = a_{(\alpha,i')}$ (up to a global constant shift absorbed into b). Thus $a_{\mathbf{r}}$ depends only on α, and $b_{\mathbf{c}}$ only on outer indices. □

3.3 Step 3: Intersection of Stabilizers

We intersect the constraints from all four modes:

1. $\dot{\lambda} = f_1(\alpha) + g_1(\beta\gamma\delta)$
2. $\dot{\lambda} = f_2(\beta) + g_2(\alpha\gamma\delta)$

Lemma 5. *The intersection of the four additive subspaces is the Segre Lie algebra.*

Proof. $f_1(\alpha) + g_1(\beta\gamma\delta) = f_2(\beta) + g_2(\alpha\gamma\delta)$. Apply discrete difference Δ_α: $\Delta_\alpha f_1(\alpha) = \Delta_\alpha g_2(\alpha\gamma\delta)$. LHS depends only on α. RHS depends on α, γ, δ. Thus $\Delta_\alpha g_2$ is independent of γ, δ, so $g_2(\alpha\gamma\delta) = h(\alpha) + k(\gamma\delta)$. Substituting back implies g_1 splits. Repeating for all modes yields $\dot{\lambda} = \sum a_\mu$. □

3.4 Step 4: Conclusion

We proved $\dim(T_Q\mathcal{Z}) = \dim(T_Q\mathcal{V})$. Since $\mathcal{V} \subseteq \mathcal{Z}$, $\mathcal{V}$ is a component of $\mathcal{Z}$. By the algebraic Implicit Function Theorem, for generic A, the solution set coincides with $\mathcal{V}$ locally. Extension to $\mathcal{I}_{\text{off}}$ follows from the density of $\mathcal{I}_{\text{obs}}$ and the closed nature of the variety.