Corrected proof of a representation theorem

Theorem 1. *For every integer $n \geq 0$ with $n \notin \{3, 15, 35\}$ there exist $x, y, z, u \in \mathbb{Z}_{\geq 0}$ such that*

$$n = x^2 + y^2 - z^2 - u^2, \qquad x^2 \leq n, \qquad y^2 \leq n, \qquad z^2 \leq n, \qquad u^2 \leq 1.$$

For $n = 3, 15, 35$ no such representation exists.

Proof. Write

$$n = m^2 + c, \qquad m = \lfloor \sqrt{n} \rfloor, \qquad 0 \leq c \leq 2m.$$

We distinguish cases according to c.

Case 1: $c = 0$. Then

$$n = m^2 + 0^2 - 0^2 - 0^2,$$

so the claim is immediate. Here $x = m$, hence

$$x^2 = m^2 \leq n.$$

Case 2: c odd. Write $c = 2t + 1$ with $0 \leq t \leq m - 1$. Since

$$2t + 1 = (t + 1)^2 - t^2,$$

we obtain

$$n = m^2 + (t + 1)^2 - t^2 - 0^2.$$

Thus $u = 0$, and here $x = m$, so $x^2 = m^2 \leq n$. Also $y = t + 1 \leq m$ and $z = t \leq m - 1$, hence

$$y^2 \leq m^2 \leq n, \qquad z^2 \leq m^2 \leq n.$$

Case 3: c even and $2 \leq c \leq 2m - 2$. Write $c = 2t$ with $1 \leq t \leq m - 1$. Since

$$2t = (t + 1)^2 - t^2 - 1,$$

we get

$$n = m^2 + (t + 1)^2 - t^2 - 1^2.$$

Hence $u = 1$, and here $x = m$, so $x^2 = m^2 \leq n$. Again $y = t + 1 \leq m$ and $z = t \leq m - 1$, so

$$y^2 \leq m^2 \leq n, \qquad z^2 \leq m^2 \leq n.$$

After Cases 1–3 only the boundary case $c = 2m$ remains, i.e.

$$n = m^2 + 2m.$$

Case 4: $c = 2m$ and m even. Then

$$2m = \left(\frac{m}{2} + 1\right)^2 - \left(\frac{m}{2} - 1\right)^2,$$

so

$$n = m^2 + \left(\frac{m}{2} + 1\right)^2 - \left(\frac{m}{2} - 1\right)^2 - 0^2.$$

Thus $u = 0$. Here $x = m$, so $x^2 = m^2 \le n$. Since $m \ge 2$ in this case, we have

$$\frac{m}{2} + 1 \le m, \qquad \frac{m}{2} - 1 \le m,$$

and therefore

$$y^2 = \left(\frac{m}{2} + 1\right)^2 \le m^2 \le n, \qquad z^2 = \left(\frac{m}{2} - 1\right)^2 \le m^2 \le n.$$

Case 5: $c = 2m$, m odd, and $m \equiv 1 \pmod 3$. If $m = 1$, then $n = 3$, which will later be shown to be impossible. Assume now $m \ge 7$. Since

$$2m + 1 = \left(\frac{m+5}{3}\right)^2 - \left(\frac{m-4}{3}\right)^2,$$

we get

$$n = m^2 + \left(\frac{m+5}{3}\right)^2 - \left(\frac{m-4}{3}\right)^2 - 1^2.$$

Hence $u = 1$. Here $x = m$, so $x^2 = m^2 \le n$. Moreover,

$$\frac{m+5}{3} \le m \qquad (m \ge 3),$$

so $y^2 \le m^2 \le n$, and likewise

$$z^2 = \left(\frac{m-4}{3}\right)^2 \le m^2 \le n.$$

Case 6: $c = 2m$, m odd, and $m \equiv 0 \pmod 3$. If $m = 3$, then $n = 15$, which will later be shown to be impossible. Assume now $m \ge 9$. Since

$$4m = \left(\frac{m+9}{3}\right)^2 - \left(\frac{m-9}{3}\right)^2,$$

we obtain

$$n = (m-1)^2 + \left(\frac{m+9}{3}\right)^2 - \left(\frac{m-9}{3}\right)^2 - 1^2.$$

Indeed,

$$(m-1)^2 + 4m - 1 = m^2 + 2m.$$

So $u = 1$. Here $x = m - 1$, hence $x^2 = (m-1)^2 < n$. Also,

$$\frac{m+9}{3} \le m \qquad (m \ge 5),$$

hence $y^2 \le m^2 \le n$, and similarly

$$z^2 = \left(\frac{m-9}{3}\right)^2 \le m^2 \le n.$$

Case 7: $c = 2m$, m odd, and $m \equiv 2 \pmod 3$. If $m \ge 17$, then

$$6m - 3 = \left(\frac{m+13}{3}\right)^2 - \left(\frac{m-14}{3}\right)^2,$$

and therefore

$$n = (m-2)^2 + \left(\frac{m+13}{3}\right)^2 - \left(\frac{m-14}{3}\right)^2 - 1^2.$$

Indeed,

$$(m-2)^2 + (6m-3) - 1 = m^2 + 2m.$$

So again $u = 1$. Here $x = m - 2$, hence $x^2 = (m-2)^2 < n$. Also

$$\frac{m+13}{3} \leq m \qquad (m \geq 7),$$

hence $y^2 \leq m^2 \leq n$, and likewise

$$z^2 = \left(\frac{m-14}{3}\right)^2 \leq m^2 \leq n.$$

The remaining odd values with $m \equiv 2 \pmod 3$ are $m = 5$ and $m = 11$. For $m = 11$ we have $n = 143$, and a direct representation is

$$143 = 9^2 + 8^2 - 1^2 - 1^2.$$

Here $x = 9$, so $x^2 = 81 < 143 = n$, and also $z^2 = 1 \leq 143 = n$. For $m = 5$ we get $n = 35$, which will later be shown to be impossible.

Combining Cases 1–7, every $n \geq 0$ except possibly $3, 15, 35$ has the required representation, with $u \in \{0, 1\}$ and in fact also $x^2 \leq n$ and $z^2 \leq n$.

It remains to show that $3, 15, 35$ are impossible.

The case $n = 3$. From $x^2 \leq 3$ and $y^2 \leq 3$ we obtain $x, y \in \{0, 1\}$, hence

$$x^2 + y^2 \leq 2.$$

But a representation would imply

$$x^2 + y^2 = 3 + z^2 + u^2 \geq 3,$$

which is impossible.

The case $n = 15$. Since $x, y \leq 3$, the possible values of $x^2 + y^2$ are

$$0, 1, 2, 4, 5, 8, 9, 10, 13, 18.$$

On the other hand, a representation with $u^2 \leq 1$ would imply

$$x^2 + y^2 = 15 + z^2 + u^2.$$

Therefore $x^2 + y^2 \geq 15$, and the only possible common value from the list above is 18. Hence one would need

$$z^2 + u^2 = 3.$$

But $u^2 \in \{0, 1\}$ and z^2 is a square, so this is impossible.

The case $n = 35$. Since $x, y \leq 5$, the possible values of $x^2 + y^2$ are

$$0, 1, 2, 4, 5, 8, 9, 10, 13, 16, 17, 18, 20, 25, 26, 29, 32, 34, 41, 50.$$

A representation with $u^2 \leq 1$ would imply

$$x^2 + y^2 = 35 + z^2 + u^2.$$

Because $x^2 + y^2 \leq 50$, we must have $z^2 + u^2 \leq 15$. Since $z^2 \in \{0, 1, 4, 9\}$ and $u^2 \in \{0, 1\}$, this gives

$$z^2 + u^2 \in \{0, 1, 2, 4, 5, 9, 10\}.$$

So necessarily

$$x^2 + y^2 \in \{35, 36, 37, 39, 40, 44, 45\}.$$

None of these numbers occurs in the above list of possible sums of two squares. This contradiction shows that 35 is impossible.

The theorem follows. $\square$