On Selection Problems with Several Non-Perfect Experts

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Contents

Abstract

In work and private life we have to make decisions every day - knowingly or unknowingly. But in some cases the process of decision making can be very complicated and difficult. For that reason, consulting more than one decider has proved useful in practice. But, are decisions made by several deciders better than decisions made by a single decider in general?

In this thesis we investigate six types of selection problems with two or more **experts**. In each of these selection problems the basic task of the experts is to select k out of n given items with maximal values or deleting $n - k$ out of n items with minimal values. These experts are not perfect, but the quality of their selected items is better than those of randomly selected items. Because of **not allowing any exchange of information** between the acting experts, they are totally independent and observe their own preference orders only.

Considering two experts A and B with equal noise levels selecting k out of n items, we focus on the question, whether the items selected by A and B are better then the items selected by A only. It shows that each double-expert-scenario with at least one action of expert B is better than letting A select all k items. Moreover, we observed interesting structures within the rankings of all these double-expert-scenarios.

Considering experts with different noise levels, we are interested in the best and worst selection orders. For alternately acting experts we achieve the best results if the worst expert selects the first item, the second worst expert the second item, . . . , and the best expert the kth item. To get the worst results, the experts should act in reverse order.

Zusammenfassung

Im beruflichen und privaten Alltag treffen wir täglich verschiedene Entscheidungen - bewusst oder unbewusst. Dabei ist das Treffen von Entscheidungen teilweise sehr komplex und nicht immer einfach. Aus diesem Grund werden in der Praxis häufig mehrere Entscheider zu Rate gezogen. Sei es bei der Vergabe von Stipendien an geeignete Bewerber oder das Packen einer Tasche für einen Ausflug. In beiden Fällen ist das Ziel, aus einer gegebenen Alternativenmenge ein oder mehrere, möglichst gute Objekte auszuwählen. Aber sind die durch mehrere Entscheider ausgewählten Alternativen tatsächlich besser als die eines einzelnen Entscheiders?

In dieser Arbeit untersuchen wir verschiedene Auswahlprobleme mit mindestens zwei Experten. Die Aufgabe der Experten besteht darin, aus n möglichen Alternativen (folgend Gegenstände genannt) k möglichst gute Gegenstände auszuwählen oder $n-k$ möglichst schlechte Gegenstände auszuschließen. Dabei sind die entscheidenden Experten nicht perfekt, aber ihre Auswahl führt zu besseren Ergebnissen als eine zufällige Auswahl. Zudem ist kein Informationsaustausch zwischen diesen Experten erlaubt, so dass jeder Experte tatsächlich nur seine eigene Präferenzordnung kennt. Mit dieser Vereinbarung sind die Experten also völlig unabhängig voneinander. Bei der Untersuchung der verschiedenen Auswahlprobleme konzentrieren wir uns auf zwei wesentliche Fragestellungen:

1. Liefern Mehr-Experten-Szenarien in der Regel bessere Ergebnisse als Ein-Experten-Szenarien?
2. Welches sind die besten Einsatz-Reihenfolgen bei Mehr-Experten-Szenarien mit unterschiedlich guten Experten?

Dazu definieren wir in Kapitel 2 sieben verschiedene Modelle für charakteristische Auswahlprobleme. In Abschnitt 2.1 beschreiben wir ein Auswahlproblem mit zwei Experten A und B ("The Selection Problem with two experts", kurz 2-SeP). Hier wählen A und B k aus n Gegenständen mit möglichst hoher Güte. Dafür definieren wir verschiedene Ein- und Mehr-Experten-Szenarien, bei denen beide Experten jeweils strikt abwechselnd agieren. Erweitert wird 2-SeP durch die Einführung zusätzlicher Experten. Im Modell "k-SeP" wählen k Experten insgesamt k Gegenstände aus, das heißt, jeder Experte darf genau einen Gegenstand wählen. Anschließend definieren wir das Model "TeSeP", wo aus drei Experten A, B und C zwei Teams gebildet werden. Diese Teams treten nun gegeneinander an, indem jedes Team abwechselnd je einen Gegenstand auswählt. Innerhalb des Zweier-Teams wird ebenfalls abwechselnd agiert. Der Wettstreit endet, sobald jedes Team k Gegenstände ausgewählt hat und Sieger ist das Team mit den insgesamt besseren Gegenständen. Zuletzt beschreiben wir drei verschiedene "GSeP"-Modelle mit zwei, drei und vier, unterschiedlich guten Experten. Diese Experten bilden je zwei Gruppen bestehend aus ein, zwei oder drei Experten. Im Gegensatz zu TeSeP führen beide Gruppen hier

unabhängig voneinander je einen Auswahl-Prozess (k aus n Gegenständen) durch. Bei vier Experten betrachten wir beispielsweise das Szenario $A;BCD$, bestehend aus einer Einer-Gruppe und einer Dreier-Gruppe. Gemäß der Szenarien-Notation wählen die Experten B, C und D strikt abwechselnd in dieser Reihenfolge.

In Kapitel 3 geben wir Einblicke in die Forschungsergebnisse von artverwandten Problemen und Fragestellungen. Während Kolassa ([Kol2004a] und [Kol2004b]) und Bärthel ([Bae2011]) ebenfalls Modelle mit unabhängigen Experten betrachteten, untersuchten Kupfer (geborene Kästner, [Kae2010]) und Hilbert ([Hil2010]) Modelle mit korrelierten Experten. In all diesen Untersuchungen zeigten sich hinsichtlich der zu Beginn dieses Kapitels beschriebenen Fragestellungen konträre Ergebnisse.

In den Kapiteln 4, 5, 6 und 7 präsentieren wir experimentelle und theoretische Ergebnisse zu allen vorgestellten Auswahlproblemen. Dabei liegt das Hauptaugenmerk auf den experimentellen Untersuchgen zu 2-SeP in Kapitel 4 und den theoretischen Untersuchungen zu GSeP in Kapitel 7.

Bei der Untersuchung von 2-SeP mit **gleich guten** Experten zeigt sich, dass jedes Zwei-Experten-Szenario besser ist als das Ergebnis des Ein-Experten-Szenarios. Der Vergleich des reinen Wahl-Szenarios AB (A und B wählen abwechselnd insgesamt k Gegenstände aus) mit dem reinen Streich-Szenarios ab (A und B streichen abwechselnd insgesamt $n-k$ Gegenstände) führte zu einem weiteren, interessanten Effekt: Für $k < \frac{n}{2}$ ist Szenario ab besser als AB und für $k > \frac{n}{2}$ ist Szenario ab schlechter als AB. Beide Szenarien sind gleich gut für $k = \frac{n}{2}$. Auch für alle weiteren Paare komplementärer Szenarien (Ab, aB) und $(ABab, abAB)$ bestätigte sich, dass Szenarien beginnend mit einer Streich-Aktion für $k < \frac{n}{2}$ bessere und für $k > \frac{n}{2}$ schlechtere Ergebnisse liefern als deren mit einer Wahl-Aktion beginnenden Komplementär-Szenarien. Zur vertiefenden Untersuchung reiner Wahl-Szenarien heben wir die bisherige Forderung $\#A = \#B = \frac{k}{2}$ (A und B führen gleich viele Wahl-Aktionen durch) auf und betrachten nun alle Wahl-Szenarien mit $\#A + \#B = k$. Es zeigt sich, dass jedes Szenario mit mindestens einer B-Aktion besser ist als die alleinige Wahl durch A. Darüber hinaus stellen wir fest, dass Szenario AB in keinem Fall bestes Szenario ist, das heißt, es gibt immer mindestens ein (gewöhnlich mehr als ein) besseres Szenario.

Sind A und B **unterschiedlich gut** (o.B.d.A. B schlechter als A), sehen wir, dass die besseren Ergebnisse erzielt werden, wenn B ein Wahl- beziehungsweise Streich-Szenario eröffnet. Im Fall von reinen Wahl-Szenarien mit abwechselnder Zugreihenfolge ist somit Szenario BA besser als Szenario AB. Dieser Effekt ist auch für ab, $ABab$ und $abAB$ und die zugehörigen B-startenden Szenarien sichtbar. Davon abweichende Ergebnisse zeigen sich bei Ab und aB, die wir kritisch diskutieren. Zudem stellen wir überrascht fest, dass Zwei-Experten-Szenarien sogar besser als die alleinige Wahl durch A sein **können**, obwohl B deutlich schlechter ist als A.

Die Einführung zusätzlicher Experten im Modell k-SeP ermöglicht es uns, die Strukturen der besten und schlechtesten Einsatz-Reihenfolgen genauer zu analysieren. Da jeder Experte genau einen Gegenstand auswählen darf, untersuchen wir für kleine k alle $k!$ Einsatz-Reihenfolgen. Dabei ist die beste Reihenfolge stets: der schlechteste Experte wählt den ersten Gegenstand, der zweitschlechteste Experte wählt den zweiten Gegenstand, ..., der beste Experte zieht den k-ten Gegenstand. Um das schlechteste Ergebnis zu erreichen, sollten die Experten genau in umgekehrter Reihenfolge eingesetzt werden. Konträr zu 2-SeP und k-SeP verhält es sich bei der besten beziehungsweise der besseren Einsatz-Reihenfolge bei TeSeP. Hier ist es vorteilhaft, innerhalb des Zweier-Teams den besseren Experten mit der Auswahl der Gegenstände beginnen zu lassen. Das heißt, mit Szenario $AB : C$ (Einsatz-Reihenfolge: $ACBC$) ist das Ergebnis des Zweier-Teams AB besser als mit Szenario $BA : C$ (Einsatz-Reihenfole: $BCAC$).

Wie schon bei 2-SeP und k-SeP, zeigt sich bei der Untersuchung von 3- und 4-GSeP mit unterschiedlich guten Experten, dass innerhalb der Mehr-Experten-Gruppen die Einsatz-Reihenfolge gemäß steigender Experten-Güte stets die besten Ergebnisse liefert. Für eine Dreier-Gruppe bestehend aus den Experten A, B und C (A besser als B und B besser als C) ist somit die beste Einsatz-Reihenfolge CBA. Und auch hier ist die schlechteste Einsatz-Reihenfole ABC. Für Szenarien mit Einer- und Zweier-Gruppen beweisen wir diese Erkenntnis mithilfe eines Permutations-Modells für den Fall $n = 3$ und $k = 2$. Dabei versteht man die Präferenzordnung eines Experten als Permutation der n Gegenstände. Auf dieser Grundlage ermitteln wir die Wahrscheinlichkeit, mit der eine Einer- beziehungsweise Zweier-Gruppe bei der Auswahl von $k = 2$ aus $n = 3$ Gegenständen den besten und zweitbesten Gegenstand auswählt. Damit zeigen wir bespielsweise, dass $A; CB$ bestes und $C; AB$ schlechtestes 3-GSeP-Szenario sind.

Für 4-GSeP unterscheiden wir zwei verschiedene Szenarien-Typen: $(1;3)$- und $(2;2)$-Typ. Szenarien des $(1;3)$-Typs bestehen aus einer Einer- und einer Dreier-Gruppe und Szenarien des $(2;2)$-Typs bestehen aus zwei Zweier-Gruppen. Unsere Untersuchungen zeigen, dass sowohl das beste, als auch das schlechteste Ergebnis mit $(1;3)$-Szenarien erzielt wird. Während alle $(2;2)$-Szenarien stets mittelgute bis gute Ergebnisse erzielen, ist die Wahl des **richtigen** $(1;3)$-Szenarios sehr wichtig.

Als Abschluss dieser Arbeit geben wir einige, aus unseren Untersuchungen entstandene Ideen, Anregungen und offene Fragen als Ausgangspunkt für zukünftige Forschungstätigkeiten an.

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1 Introduction

Every day we have to make decisions - knowingly or unknowingly - regardless of whether these processes of decision making are within the scope of work or private life. For instance,

in work life:

- presenting selected approaches to a problem to the supervisor or
- awarding scholarships to the most qualified applicants

and in private life:

- buying food and non-food in the supermarket,
- packing bags for a journey, or
- watching TV in the free time.

In each of these situations we have to decide between a set of possible alternatives with resulting in one or more alternatives. Therefore, division of work or teamwork are common methods. But, are the selected alternatives resulting from these methods better than these ones resulting from a single decision maker in general?

In this thesis we investigate six types of selection problems with two or more experts. The task of these experts in the basic model "The Selection Problem with Two Experts" is to select k alternatives (= items) out of n possible items with maximal quality. The experts are not perfect, but the quality of the items selected by the experts has the distinction of being better than the quality of randomly selected items. For this purpose, each expert is assigned to a specific noise level.

In our models there is no exchange of information between any acting experts. Each expert knows his own preference order only. So, the experts considered in the models are absolutely independent. To divide the process of selecting items we will define several scenarios.

This thesis is structured as follows:

In Chapter 2 we introduce altogether six models of characteristic selection problems. Section 2.1 presents the Selection Problem with two experts (2-SeP) as mentioned previously. Here, two experts A and B select k out of n items with largest values. Therefore, we define several scenarios in which the relative amount of expert A is 1 (no actions of expert B), $\frac{1}{2}$ ($\#A$ actions $= \#B$ actions), or 0 (no actions of expert A). For the second case we consider scenarios with selection or deletion actions. In detail, to select k items with largest values we can delete $n - k$ items with smallest values instead. In Section 2.2 we extend 2-SeP by increasing the number of experts to k. So, each expert is allowed to select only one item. After that, Section 2.3 presents the Team Selection Problem (TeSeP). Here, three experts A, B, and C form two teams and compete against each other. Both teams alternately select k out of n items and the team with larger cumulative sum of their selected items wins. The last section of Chapter 2, Section 2.4, introduces three models of the Group Selection Problem (GSeP). Up to four experts form two groups including one, two, or three experts. In contrast to TeSeP, each group selects k out of n items one after the other and independent of the other group.

In Chapter 3 we give some insights of investigations relating to the Selection Problem. Among shortlisting methods analysed by Kolassa ([Kol2004a] and [Kol2004b]) and Hilbert ([Hil2010]), we summarise systems of division of work with correlated experts investigated by Kupfer, née Kästner ([Kae2010]). As one case of the Selection Problem we outline the Assortment Problem analysed by Bärthel ([Bae2011]) in the last section of this chapter.

Chapters 4, 5, 6, and 7 show experimental and theoretical results for the selection problems characterised in Chapter 2. The most important results are outlined in Chapter 4 (experimental results for 2-SeP) and Chapter 7 (experimental and theoretical results of GSeP).

Chapter 4 presents experimental results for two experts with equal and different noise levels. For both cases we investigate single- and double-expert-scenarios. Among the double-expert-scenarios defined in Section 2.1 we consider all possible selection or deletion orders of experts A and B.

In Chapter 5 we outline experimental results of the Selection Problem with k experts (k-SeP). Here, we suppose two different distributions for the noise levels of the experts. Providing these distributions, we consider all $k!$ possible selection orders of the experts and focus on specific structures within these orders.

In Chapter 6 we investigate TeSeP by using four different cases of the noise levels and describe the corresponding, experimental results.

Chapter 7 shows experimental results for all characterised Group Selection Problems for experts with different noise levels. The main part of this chapter are the theoretical results for the Group Selection Problem with three and four experts. Here, we focus on the relations between all considered scenarios.

Chapter 8 summarises the main results and observations of all models investigated. It also discusses several interesting structures that appeared.

The last chapter of this thesis, Chapter 9, gives some approaches, ideas and open questions for future work. It finishes with a short outlook.

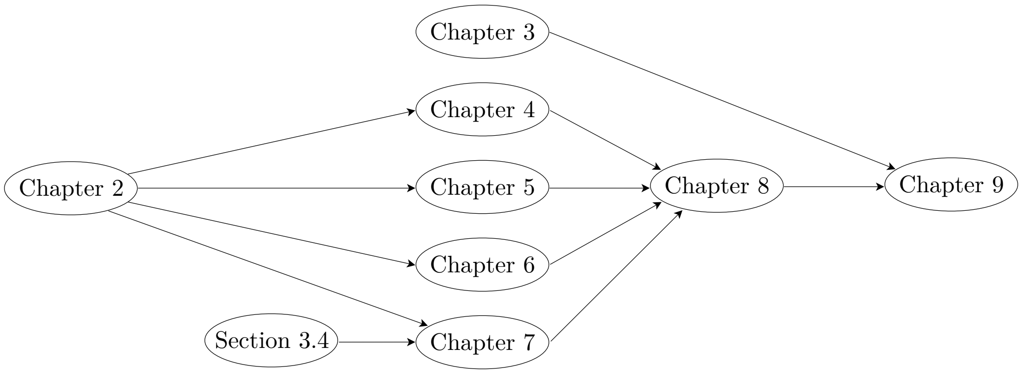


Figure 1.1 How to read this thesis.

2 The Selection Problem with Two or More Experts

In this chapter several kinds of the Selection Problem will be described. The basic task in this problem is to select a subset of items from a given set. More precisely, this subset should preferably include items with largest cumulative sum. The Selection Problem is an easy problem but the challenge is to find a good solution provided that all experts involved are not perfect.

The following sections present altogether six models of characteristic selection problems. In all of our models there is no exchange of information between any acting experts.

2.1 The Selection Problem with Two Experts

Model 2.1.1 (The Selection Problem with two experts, shortly called 2-SeP) *Consider two non-perfect experts A and B with noise levels v_a and v_b and the following independently distributed random variables:*

- *n items with true values* $x_1, x_2, \dots, x_n \sim \mathcal{N}_{0,1}$
- *n noise values for expert A* $a_1, a_2, \dots, a_n \sim \mathcal{N}_{0,v_a}$
- *n noise values for expert B* $b_1, b_2, \dots, b_n \sim \mathcal{N}_{0,v_b}$

Expert A observes $(x_1+a_1, x_2+a_2, \dots, x_n+a_n)$ and B observes $(x_1+b_1, x_2+b_2, \dots, x_n+b_n)$. We call them independent experts because the a_i and b_j are independent.

The task in 2-SeP is to construct a k-set $S_k \subseteq \{1, 2, \dots, n\}$ consisting of the indices of the $k \in \{1, 2, \dots, n\}$ largest items x_i. We call k the selection number. Therefore, we define several scenarios how two experts can construct such a set. In each scenario expert A starts. Experts A and B always act alternately. In general, capital letters represent **selection actions** and small letters stand for **deletion actions**. A scenario stops if k items are selected or $n - k$ items are deleted.

The following overview shows all considered scenarios. As mentioned before, both experts act by turns. Each scenario name describes a periodic sequence of action.

A ________	A selects the largest k items according to his values $x_i + a_i$.
AB ________	A and B select altogether k items of their largest observation.
ab ________	A and B delete altogether $n - k$ items of their smallest observation.
Ab ________	A selects items and B deletes items.
aB ________	A deletes items and B selects items.
$ABab$ _____	A and B each select an item and each delete an item.
$abAB$ _____	A and B each delete an item and each select an item.

It is not required to consider scenario "a" (A deletes his smallest $n-k$ items) because it leads to the same result as scenario A.

Definition 2.1.1 *We declare three different characteristics of expert noise.*

(i) Expert A is called perfect if $v_a = 0$.

(ii) Expert A and B are called equally good if $v_a = v_b$.

(iii) Expert A is better than B if $v_a < v_b$.

In case of (iii) not only the scenarios A, AB, ab, Ab, aB, $ABab$, and $abAB$, but also the scenarios B, BA, ba, Ba, bA, $BAba$, and $baBA$ are considered. In all these settings expert B starts.

The following example illustrates the idea of 2-SeP. This example will appear also in subsequent cases.

Example Consider $n = 8$, $k = 4$ and $x = (8, 5, 3, 2, 1, 0, -2, -2)$. The optimal solution set is $S_4^* = \{1, 2, 3, 4\}$ with total sum $w_4^* = 8 + 5 + 3 + 2 = 18$. We also consider $a = (-3, -2, -3, 0, -2, -1, 0, 1)$ and $b = (0, -1, -3, 2, 3, 2, -3, 2)$.

The Selection Problem will exemplarily be demonstrated by using the double-expert-scenario $ABab$. Ties are broken by fair coin flips.

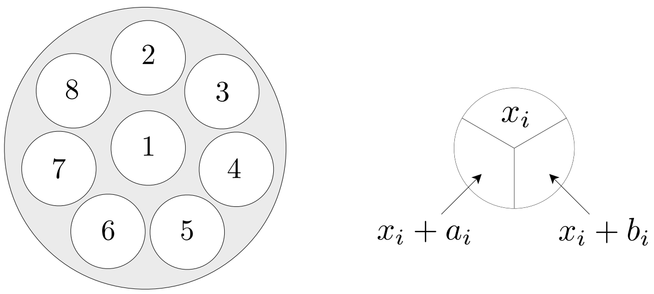
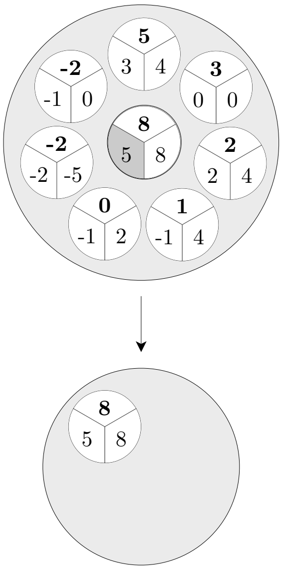
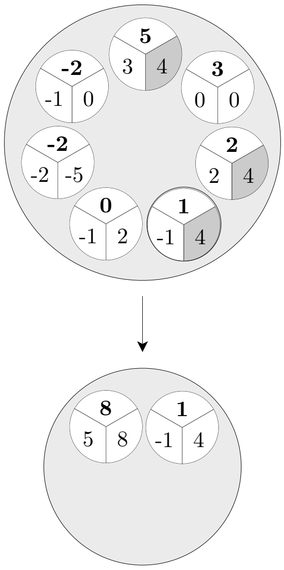


Figure 2.1 Left: Item order. Right: Item labelling.

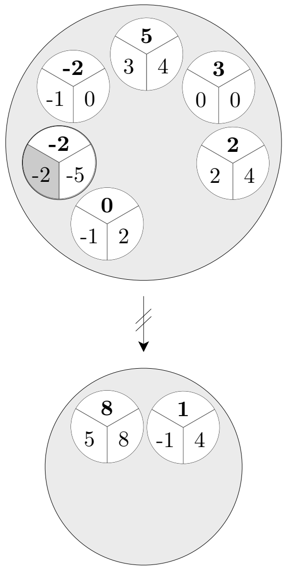
The left part of Figure 2.1 shows how these $n = 8$ items are positioned. The first item is located in the middle of the pot and the other seven items are positioned around in a clockwise direction starting at the top. We choose this order only to clearly demonstrate the procedure of the Selection Problem. Of course, experts A and B see the items ordered according to their observations. The right part of Figure 2.1 shows how each item is labelled. The upper third contains the true values and the lower third on the left (right) side contains the observed values of expert A (B) of the considered item.



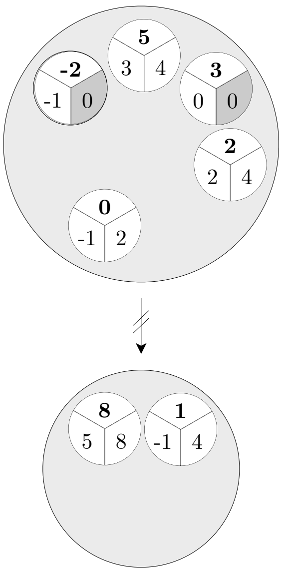
Step 1 (action "*A*"): Expert A selects item 1.



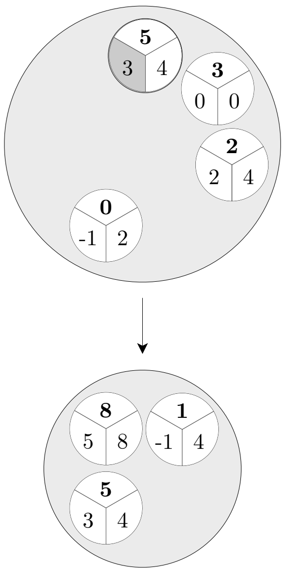
Step 2 (action "*B*"): Expert B has to decide between items 2, 4, and 5. He selects item 5.



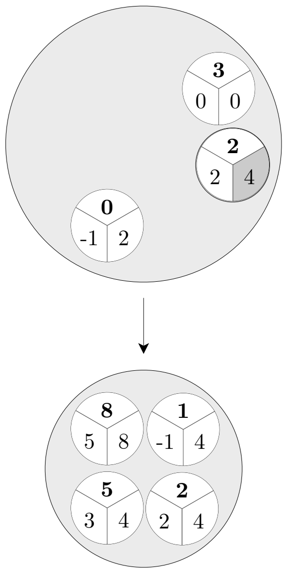
Step 3 (action "*a*"): Expert A deletes item 7.



Step 4 (action "*b*"): Expert B has to decide between items 3 and 8. He deletes item 8.



Step 5 (action "*A*"): Expert A selects item 2.



Step 6 (action "*B*"): Expert B selects item 4.

In this scenario there have to be made six steps to select altogether $k = 4$ items. The solution set is $S_4^{ABab} = \{1, 5, 2, 4\}$ with total sum $w_4^{ABab} = 8 + 1 + 5 + 2 = 16$.

2.2 The Selection Problem with k Experts

As one extension of 2-SeP we now introduce the Selection Problem with k experts. But as a constraint each expert is allowed to *select only one item*.

Model 2.2.1 (The Selection Problem with k experts, shortly called k-SeP) *Consider $k \in \{1, 2, \dots, n\}$ non-perfect and independent experts $E_1, E_2, \dots, E_k$ with noise levels $v_1 < v_2 < \dots < v_k$ and the following independently distributed random variables:*

- *n items with true values* $x_1, x_2, \dots, x_n \sim \mathcal{N}_{0,1}$
- *n noise values for expert E_1* $e_{1,1}, e_{1,2}, \dots, e_{1,n} \sim \mathcal{N}_{0,v_1}$

  $\vdots$
- *n noise values for expert E_k* $e_{k,1}, e_{k,2}, \dots, e_{k,n} \sim \mathcal{N}_{0,v_k}$

Expert E_j $(j = 1, \dots, k)$ observes the set $\{x_i + e_{j,i} : i = 1, \dots, n\}$.

The target in k-SeP is to get a picking order in which the cumulative sum of the selected items is as large as possible. Therefore, we consider all $k!$ different picking orders in which every expert selects exactly one item per order.

2.3 The Team Selection Problem

In this section we deal with three experts forming two teams and these teams compete against each other.

Model 2.3.1 (The Team Selection Problem, shortly called TeSeP) *Consider three non-perfect and independent experts A, B and C with noise levels v_a, v_b and v_c and the following independently distributed random variables:*

- *n items with true values* $x_1, x_2, \dots, x_n \sim \mathcal{N}_{0,1}$
- *n noise values for expert A* $a_1, a_2, \dots, a_n \sim \mathcal{N}_{0,v_a}$
- *n noise values for expert B* $b_1, b_2, \dots, b_n \sim \mathcal{N}_{0,v_b}$
- *n noise values for expert C* $c_1, c_2, \dots, c_n \sim \mathcal{N}_{0,v_c}$

Expert A observes the set $\{x_i + a_i : i = 1, \dots, n\}$, B observes the set $\{x_i + b_i : i = 1, \dots, n\}$ and C observes the set $\{x_i + c_i : i = 1, \dots, n\}$.

In contrast to the other two models introduced until now, experts A and B work together and compete against expert C. Now each team has to construct its own k-set $S_k^{AB}, S_k^C \subseteq, \{1, 2, \dots, n\}$ with $S_k^{AB} \cap S_k^C = \emptyset$ consisting of the indices of the $k \in \{1, 2, \dots, \frac{n}{2}\}$ largest items x_i. So, each team selects k items. Both team AB and team C select alternately. Experts A and B act alternately as well. The team with larger cumulative sum of its selected items wins.

Referring to the notation in sporting competitions we use a colon to underline that both teams compete against each other. A short overview shows all considered scenarios.

$AB : C$ ___	A and B compete against C. The periodic picking order is $ACBC$.
$C : AB$ ___	C competes against A and B. The periodic picking order is $CACB$.

If experts A and B are not equally good (see Definition 2.1.1 (iii) on page 18), we consider scenario $BA : C$ and $C : BA$ as well. The periodic picking orders are $BCAC$ and $CBCA$.

Example Consider $n = 8$, $k = 2$ and $x = (8, 5, 3, 2, 1, 0, -2, -2)$. The optimal solution set is $S_4^* = \{1, 2, 3, 4\}$ with total sum $w_4^* = 8 + 5 + 3 + 2 = 18$. We also consider $a = (-3, -2, -3, 0, -2, -1, 0, 1)$, $b = (0, -1, -3, 2, 3, 2, -3, 2)$, and $c = (1, 0, 1, 0, -1, -2, -1, -1)$.

The idea of TeSeP will exemplarily be demonstrated by using the scenarios $AB : C$ (team AB starts) and $C : AB$ (team C starts). Again, ties are broken by fair coin flips.

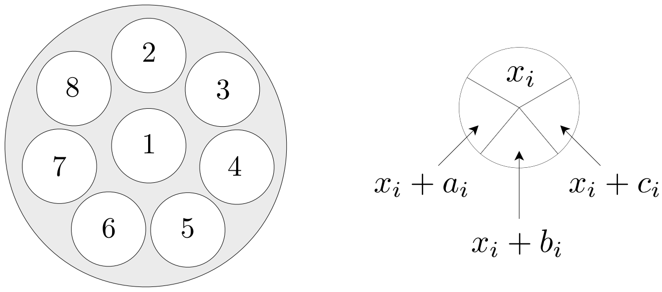
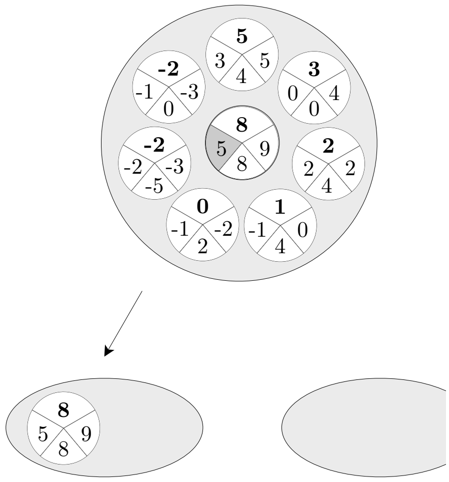
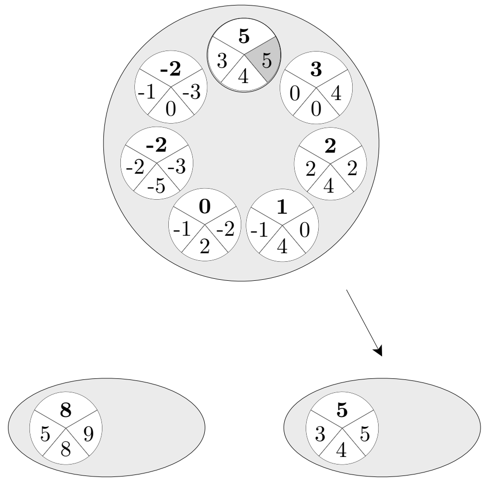


Figure 2.2 Left: Item order. Right: Item labelling.

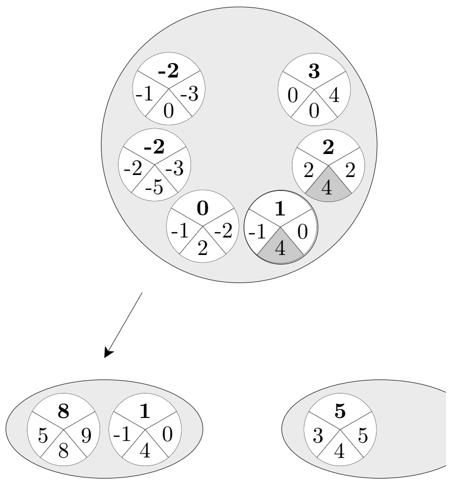
As described in Figure 2.1 on page 18 the figure above illustrates how the items are positioned and labelled.



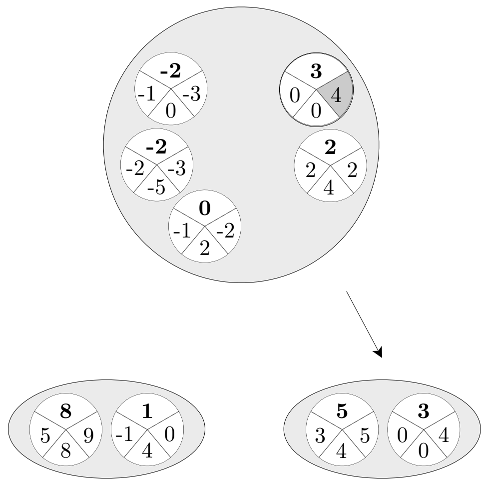
Step 1 (action "A"): Expert A selects item 1.



Step 2 (action "C"): Expert C selects item 2.



Step 3 (action "B"): Expert B has to decide between items 4 and 5. He selects item 5.



Step 4 (action "C"): Expert C selects item 2.

Team AB wins by $9 : 8$ if they start selecting in the introduced picking order (scenario $AB : C$). On the other hand, the single-expert-team C wins by $11 : 7$ if he starts selecting (scenario $C : AB$). The second result is figured out in Appendix C on page 109.

2.4 The Group Selection Problem

The Group Selection Problem (shortly called GSeP) deals with up to four experts forming two groups including one, two, or three members. In contrast to the other models introduced, we only allow experts with different noise levels v. Furthermore, both groups select their items one after the other. So, the first group selects its items and then the true values of these items are recorded. Afterwards we put the selected items back to the other ones and then the second group selects its items. Their true values are recorded as well. Altogether we are interested in the total sum of the selected items of *both* groups.

All in all we will introduce three models differing in the number of experts. The first model describes the Group Selection Problem with two experts.

Model 2.4.1 (2-GSeP) *Consider two non-perfect and independent experts A and B with noise levels $v_a < v_b$ and the following independently distributed random variables:*

- *n items with true values* $x_1, x_2, \ldots, x_n \sim \mathcal{N}_{0,1}$
- *n noise values for expert A* $a_1, a_2, \ldots, a_n \sim \mathcal{N}_{0,v_a}$
- *n noise values for expert B* $b_1, b_2, \ldots, b_n \sim \mathcal{N}_{0,v_b}$

Expert A observes the set $\{x_i + a_i : i = 1, \ldots, n\}$ and B observes the set $\{x_i + b_i : i = 1, \ldots, n\}$.

As introduced above experts A and B will be assigned to two single-expert-groups. The selection numbers are k_1 and k_2 with $k_2 < k_1 \in \{1, 2, \ldots, n\}$.

In Section 2.3 on page 20 we used the colon notation to signalize the competition between the acting teams. We now use the semicolon notation because there is no competition any more. According to these agreements we consider the following two scenarios.

$A_{k_1}; B_{k_2}$ —— A selects k_1 and B selects k_2 items.
$A_{k_2}; B_{k_1}$ —— A selects k_2 and B selects k_1 items.

In the next model of GSeP we integrate one more expert C. We now consider single- and double-expert-groups. The model is as follows.

Model 2.4.2 (3-GSeP) *Consider three non-perfect and independent experts A, B, and C with noise levels $v_a < v_b < v_c$ and the following independently distributed random variables:*

- *n items with true values* $x_1, x_2, \ldots, x_n \sim \mathcal{N}_{0,1}$
- *n noise values for expert A* $a_1, a_2, \ldots, a_n \sim \mathcal{N}_{0,v_a}$
- *n noise values for expert B* $b_1, b_2, \ldots, b_n \sim \mathcal{N}_{0,v_b}$
- *n noise values for expert C* $c_1, c_2, \ldots, c_n \sim \mathcal{N}_{0,v_c}$

Expert A observes the set $\{x_i + a_i : i = 1, \ldots, n\}$, B observes the set $\{x_i + b_i : i = 1, \ldots, n\}$, and C observes the set $\{x_i + c_i : i = 1, \ldots, n\}$.

Each group selects $k \in \{1, 2, \ldots, n\}$ items. Double-expert-groups alternately select these items. We define altogether six scenarios: $A;BC$, $A;CB$, $B;AC$, $B;CA$, $C;AB$ and $C;BA$.

Now we integrate one more expert D, increasing their number to four.

Model 2.4.3 (4-GSeP) *Consider four non-perfect and independent experts A, B, C, and D with noise levels $v_a < v_b < v_c < v_d$ and the following independently distributed random variables:*

- *n items with true values* $x_1, x_2, \ldots, x_n \sim \mathcal{N}_{0,1}$
- *n noise values for expert A* $a_1, a_2, \ldots, a_n \sim \mathcal{N}_{0,v_a}$
- *n noise values for expert B* $b_1, b_2, \ldots, b_n \sim \mathcal{N}_{0,v_b}$
- *n noise values for expert C* $c_1, c_2, \ldots, c_n \sim \mathcal{N}_{0,v_c}$
- *n noise values for expert D* $d_1, d_2, \ldots, d_n \sim \mathcal{N}_{0,v_d}$

Expert A observes the set $\{x_i + a_i : i = 1, \ldots, n\}$, B observes the set $\{x_i + b_i : i = 1, \ldots, n\}$, C observes the set $\{x_i + c_i : i = 1, \ldots, n\}$, and D observes the set $\{x_i + d_i : i = 1, \ldots, n\}$.

In the same way as done in Model 2.4.2 we declare groups including one, two, and three experts. Altogether there are 12 scenarios of **(2;2)-type** (two double-expert-groups, e.g. $CA;BD$) and 24 scenarios of **(1;3)-type** (one single- and one triple-expert-group, e.g. $B;ADC$). All 36 considered scenarios are listed in Appendix D on page 111.

3 Related Work

The following four theses are inspired by the "3-Hirn-Prinzip" introduced by Ingo Althöfer in his article "Das Dreihirn – Entscheidungsteilung im Schach" published in 1985 (in German, [Alt1985]) and his book "13 Jahre 3-Hirn – Meine Schach-Experimente mit Mensch-Maschinen-Kombinationen" published in 1998 (in German, [Alt1998]). The 3-Hirn-Prinzip was firstly tested in chess games. One person and two computers are involved. Each computer calculates a possible solution and the person decides for one of these proposed solutions. So, with **one** human brain and **two** computer-aided brains there are **three** brains ("3-Hirn") in total.

3.1 Multi-Step Shortlisting by Imperfect Experts

In 2004, Stephan Kolassa ([Kol2004b]) published his doctoral dissertation "Multi-Step Shortlisting by Imperfect Experts". The main focus of his work is the investigation of the advantages and disadvantages of multi-step decisions between several alternatives by imperfect experts. In this section double-step shortlisting as compared to single-step decisions will be the central theme. The experts are called Alice and Bob.

To be exact, the target of Alice and Bob is to find the best (= maximum quality) alternative out of n possible alternatives. In the first step, Alice selects the indices of her $k \le n$ largest observations into a so-called "shortlist". For this purpose, Kolassa called k the shortlist size. In the second step, Bob selects his largest observation among the shortlisted alternatives. So, the basic model is described as follows:

Model 3.1.1 (Model Conti) *Consider $3n$ independently distributed random variables*

$$x_1, \dots, x_n \text{ uniformly distributed in } [0,1],$$

$$\gamma_1, \dots, \gamma_n \text{ uniformly distributed in } [0,\Gamma], \text{ and}$$

$$\delta_1, \dots, \delta_n \text{ uniformly distributed in } [0,\Delta].$$

We consider two experts, Alice and Bob. Alice observes the set $\{x_i + \gamma_i : 1 \le i \le n\}$ and constructs a shortlist $S \subseteq \{1, \dots, n\}$ consisting of the indices of her largest observations $x_i + \gamma_i$. Bob now observes the set $\{x_i + \delta_i : i \in S\}$ and selects the index $\tilde{i}$ of his largest observation. The correct index i^* Alice and Bob are looking for is the index of the maximal x_i.

The x_i quantify the true value of alternative i and the γ_i and δ_i present the noise values of Alice and Bob. So, increasing Γ and Δ decreases the precision of the experts. Kolassa mostly restrict his attention to equally good experts Alice and Bob

($\Gamma = \Delta$). He called the probability of returning the index i^* of the "real" maximum $\{\max x_i : 1 \le i \le n\}$ the hitting ratio

$$h_{Conti}(n \xrightarrow{\Gamma} k \xrightarrow{\Delta} 1).$$

For his experimental investigations he conducted Monte Carlo simulations and defined the empirical hitting ratio as

$$h_{Conti}^{emp} := \frac{\#\{\text{Monte Carlo runs in which } i^* \text{ is returned}\}}{\#\{\text{all Monte Carlo runs}\}}.$$

After explaining the basic model of multi-step shortlisting we will give certain experimental results analysed by Kolassa ([Kol2004b]) now.

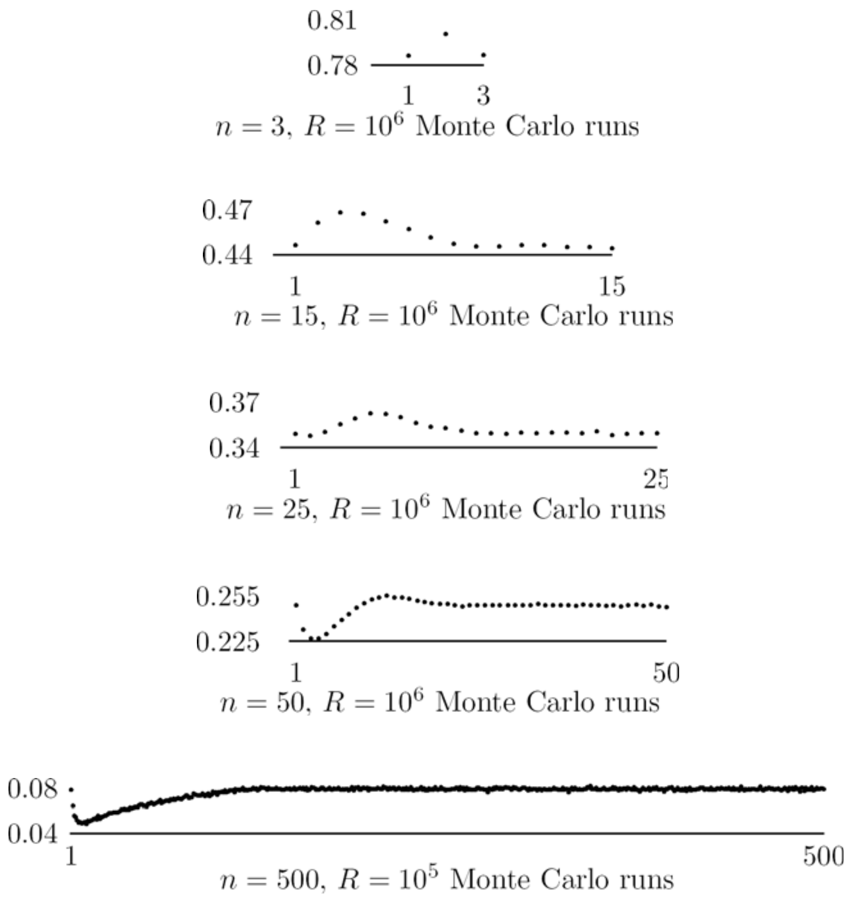


Figure 3.1.1 Empirical hitting ratios for $n \in \{3, 15, 25, 50, 500\}$ as a function of k in Model Conti. The noise levels are $\Gamma = \Delta = \frac{1}{2}$. Adapted from [Kol2004b] with permission by Kolassa.

Obviously, for $n \le 15$ the empirical hitting ratio is unimodal. By increasing the shortlist size k it first increases and then decreases. For example, for $n = 3$ there are three cases:

$3 \xrightarrow{\Gamma} 1 \xrightarrow{\Delta} 1$	is worse than	$3 \xrightarrow{\Gamma} 2 \xrightarrow{\Delta} 1$	is better than	$3 \xrightarrow{\Gamma} 3 \xrightarrow{\Delta} 1$
single-step decision of Alice		“real” double-step decision		single-step decision of Bob

So, the double-step decision ($k = 2$) is better than both single-step decisions of Alice or Bob ($k \in \{1, 3\}$). For $n \geq 25$ the empirical hitting ratio first decreases by increasing k. So, for all these n the single-step decision of Alice is better than the double-step decision for $k = 2$. By increasing n the number of different shortlist size k occurring this phenomenon increases. Kolassa called this phenomenon a shortlisting valley. To investigate the presence or absence of shortlisting valleys he also analysed normally and exponentially distributed true and noise values. Among valleys there are also other (topographical) structures like “peaks” and “tables”. Especially, for normally distributed true and noise values there are no valleys. The results for all three distributions can be found in his doctoral dissertation ([Kol2004b]) and in the technical report ([Kol2004a]).

3.2 Optimisation Problems of Sum Type with Correlated Experts

First investigations regarding division of work by Nancy Kupfer, née Kästner, were published in 2010 in the diploma thesis “Summentyp-Optimierungs-Probleme mit korrelierten Experten” (in German, [Kae2010]). In this thesis we analysed five different optimisation problems of sum type: the Shortest Path Problem in Weighted, Directed Grid Graphs (WDGG), the Minimum Spanning Tree Problem (MST), The Assignment Problem (AP), the Knapsack Problem (KP), and the Travelling Salesperson Problem (TSP). In contrast to the selection problems introduced in Chapter 2 starting on page 17 all of them are minimisation problems.

Analogously to Model 2.1.1 on page 17 we defined independently distributed random variables $x_1, x_2, \ldots, x_n$ (= true values), $a_1, a_2, \ldots, a_n$ (= noise values for expert A), and $b_1, b_2, \ldots, b_n$ (= noise values for expert B). Moreover, we introduced common noise values for expert A and expert B ($c_1, c_2, \ldots, c_n$). With these restrictions we described the following three models without exchange of information between both experts:

		A observes	B observes
(i)	independent experts (indep)	$x_i + a_i$	$x_i + b_i$
(ii)	positively correlated experts (pos-corr)	$x_i + a_i + c_i$	$x_i + b_i + c_i$
(iii)	negatively correlated experts (neg-corr)	$x_i + a_i + c_i$	$x_i + b_i - c_i$

Moreover, we specified three scenarios dividing the process of solving each of the optimisation problems listed above:

A ———	Expert A solves the optimisation problem all alone.
AB ———	Expert A solves the first half and expert B solves the second half of the optimisation problem.
$abab$ ———	Experts A and B solve the optimisation problem alternately step by step.

Remark With the notations introduced in Chapter 2.1 on page 18 the last scenario listed would be called AB. But due to AB is already used for the second scenario listed, we use the notation defined in the diploma thesis ([Kae2010]).

To show which model and which scenario is best or worse we present experimental results for WDGG and MST-P (MST solved using Prim's algorithm [Pri1957]). The true and noise values are uniformly distributed as declared in the caption of Figure 3.2.1. The ordinate represents the average, absolute performance differences between the optimal solution and the solutions of the considered models and scenarios.

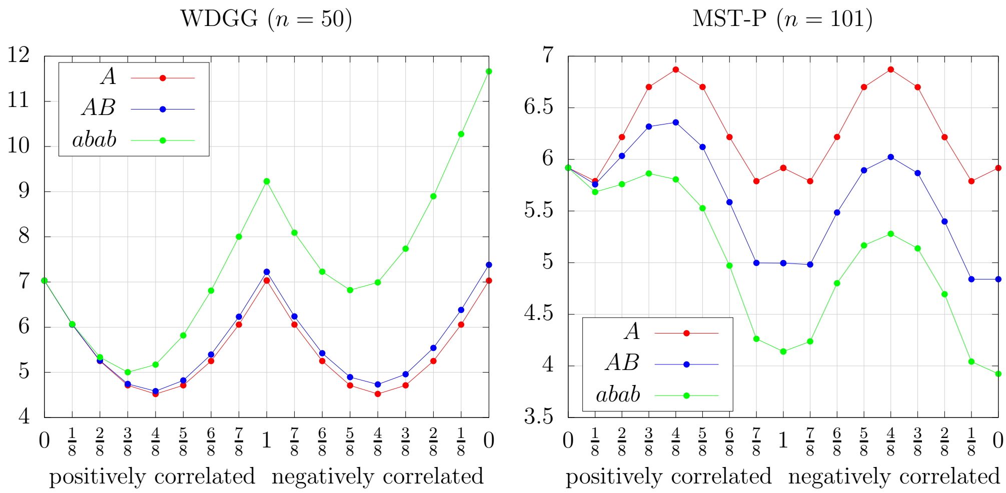


Figure 3.2.1 WDGG and MST-P for $x_i \sim U(0,1)$, $a_i \sim U(0,\alpha)$ and $b_i \sim U(0,\beta)$ with $\alpha = \beta \in \{\frac{i}{8} : i = 1,2,\ldots,8\}$ (abscissa), $c_i \sim U(0,\gamma)$ with $\gamma = 1-\alpha = 1-\beta$, and $T = 10^6$ runs. Taken from [Kae2010].

Obviously, the quality of the models and scenarios depends on the optimisation problem itself. The results are totally contrary. On the one hand, for WDGG

- A is slightly better than AB is better than $abab$ ($A \lessapprox AB < abab$) and
- positively correlated experts are better than negatively correlated experts.

And on the other hand, for MST-P

- A is worse than AB is worse than $abab$ ($A > AB > abab$) and
- positively correlated experts are worse than negatively correlated experts.

Furthermore, for each scenario of WDGG positively correlated experts are better than independent experts ($\alpha = \beta = 1$ and $\gamma = 0$). In our diploma thesis we investigated two different distributions for the true and noise values: uniform and normal distribution. Whereas the relations of models (*ii*) and (*iii*) stay constant dependent on the distribution for each optimisation problem, the relations of the scenarios vary. Table 3.2.1 and Figure 3.2.2 give a compact overview about these relations. MST was solved using Prim's [Pri1957] or Kruskal's [Kru1956] algorithm (MST-P or MST-K). AP, KP, and TSP were solved using fast or complete local search (FLS or CLS) with a heuristic initial solution.

pos-corr < neg-corr	pos-corr > neg-corr
WDGG	
	MST-K and MST-P
AP-FLS	AP-CLS
	KP-FLS and KP-CLS
TSP-FLS	TSP-CLS

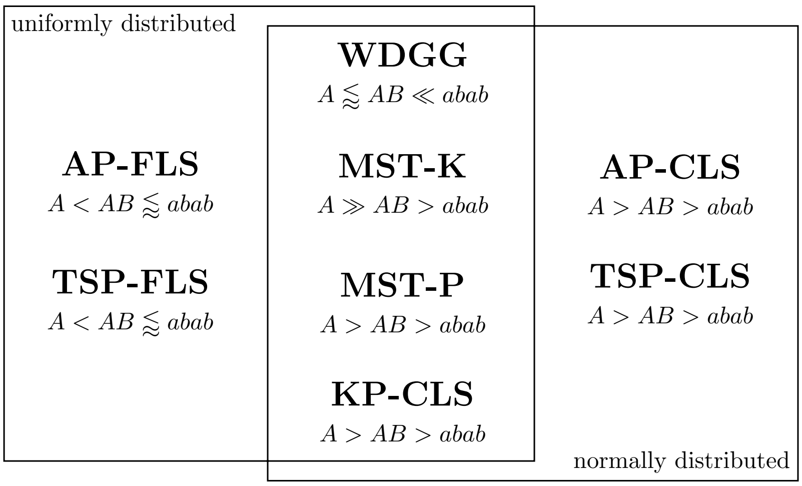


Figure 3.2.2 Relations between uniform and normal distribution for all considered optimisation problems. Taken from [Kae2010].

For KP-FLS the scenarios are not relatable in this way. Summarising the results presented previously, none of the scenarios defined is best for all (considered) optimisation problems. Moreover, negatively correlated experts are more often better than positively correlated experts. Additional and more detailed information can be found in the diploma thesis ([Kae2010]).

3.3 Several Selection Methods with Correlated Experts

Based on the investigations of multi-step shortlisting by Kolassa (see Section 3.1 starting on page 25, [Kol2004b]), Anne Hilbert analysed several selection methods in her diploma thesis "Auswahlverfahren mit korrelierten Experten" (in German, [Hil2010]). To find the best alternative out of n possible alternatives she investigated altogether seven selection methods. Within these methods there are two shortlisting methods:

- "Shortlisting" $n \xrightarrow{\text{expert A}} k \xrightarrow{\text{expert B}} 1$
- "Double-Shortlisting" $n \xrightarrow{\text{expert A}} k_1 \xrightarrow{\text{expert B}} k_2 \xrightarrow{\text{expert A}} 1$

Remark Considering the notation introduced by Kolassa ([Kol2004b]), the first shortlisting method listed above would be called "double-step shortlisting" and the second shortlisting method listed above would be called "triple-step shortlisting". Because of summarising Hilbert's work, we use the notation introduced by her.

Hilbert defined the true values and the noise values for expert A and expert B in the same way as done in Model 2.1.1 on page 17. Analogously to the models defined by Kupfer (see Section 3.2 on page 27, [Kae2010]), she describes three models for independent, positively correlated, and negatively correlated experts A and B. Now we present experimental results of Shortlisting and Double-Shortlisting for these three models analysed by Hilbert in her thesis. The ordinate represents the empirically expected result

$$\tilde{x}^{emp} := \frac{\sum_{m=1}^{t} \tilde{x}_m}{t}$$

of both methods based on $t = 10^6$ Monte Carlo runs. $\tilde{x}_m$ means the result of the mth run. It depends on n and k or k_1 and k_2.

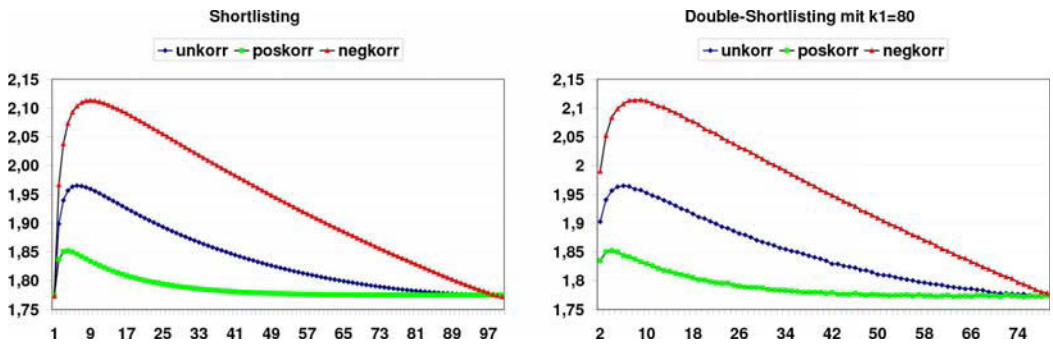


Figure 3.3.1 Shortlisting and Double-Shortlisting with independent (blue), positively correlated (green), and negatively correlated (red) experts for $n = 100$, $k \in \{1, 2, \dots, 100\}$ (abscissa left plot), $k_1 = 80$ and $k_2 \in \{2, 3, \dots, 79\}$ (abscissa right plot), $x_i \sim \mathcal{N}_{0,1}$, $a_i, b_i, c_i \sim \mathcal{N}_{0,\frac{1}{2}}$, and $t = 10^6$ runs. In the diagrams, large values are good. Taken from [Hil2010] with permission by Hilbert.

Obviously, for both shortlisting methods the results with negatively correlated experts are better than the results with independent or positively correlated experts.

In contrast to Kolassa's experimental results for uniformly distributed true and noise values there are no shortlisting valleys. In addition to these two shortlisting methods, Hilbert defined five more methods of finding the best alternative out of n possible alternatives. Four of them are deletion procedures where experts A and B delete "worse" alternatives step by step until one alternative is left over. The last method Hilbert introduced in her thesis ([Hil2010]) is the "Borda-Verfahren" (in english: Borda count, [Bor1781], [Gra1953]). Here, expert A ranks the list of possible alternatives in order of decreasing quality according to his observation. After that he gives the mth worst alternative m points. So, the best (= nth worst) alternative gets n points, the $(n-1)$th worst alternative gets $n-1$ points, ..., until the worst alternative gets one point. Expert B conducts the same procedure according to his observation. After that the points of A and B are added for each alternative i ($i = 1, 2, \dots, n$) and the result of Borda count is the alternative with highest number of points. If there exists more than one alternative with highest number of points, Hilbert selected randomly, uniformly distributed.

For all seven considered methods the results with negatively correlated experts are better than the results with independent or positively correlated experts. Furthermore, the **best**, empirically expected results of each shortlisting and each deletion method are equal. And, they are all worse than the results of Borda count.

n	Shortlisting	Borda Count
5	0.93	0.96
10	1.26	1.30
25	1.63	1.69
50	1.88	1.94
100	2.11	2.17
500	2.59	2.64

Table 3.3.1 Best, empirically expected results for Borda Count and Shortlisting with negatively correlated experts. Taken from [Hil2010] with permission by Hilbert.

Table 3.3.1 includes detailed results for Borda count and Shortlisting (exemplary for all equally good methods) with negatively correlated experts. The definitions of the models and additional results for the methods not mentioned by name can be found in Hilbert's diploma thesis ([Hil2010]).

3.4 The Assortment Problem

As one case of the Selection Problem (see Chapter 2 starting on page 17) the Assortment Problem investigated by Marlis Bärthel in her report "Viele Köche verderben nicht immer den Brei - Ein diskretes Expertenmodell zum Assortment-Problem" (in German, [Bae2011]) describes two and three non-perfect experts selecting k out of n possible alternatives.

Consider n alternatives with values $x_1 < x_2 < \ldots < x_n \in \mathbb{R}$. The task in the Assortment Problem is to select k out of n alternatives where the sum of their values is as small as possible. Similar to Model 2.1.1 elucidated on page 17 there are two non-perfect experts A and B. So, A observes $\{x_i + a_i : a_i \in \mathbb{R}, i = 1, 2, \ldots, n\}$ and B observes $\{x_i + b_i : b_i \in \mathbb{R}, i = 1, 2, \ldots, n\}$. Altogether, she analysed three scenarios: A, AB, and BA (see Section 2.1 starting on page 17).

To understand the observations of expert A and expert B as permutations $\pi \in S_n$ she redefined the values of the alternatives. In detail, $1 := x_1,\ 2 := x_2,\ \ldots,\ n := x_n$. So, a permutation represents the order in which an expert ranks the alternatives. To give an example, consider $n = 3$ and $\pi = (2, 3, 1)$. Here, the expert believes in $x_2 < x_3 < x_1$. Asking him to select the two smallest alternatives he would select the second best and the third best alternatives.

In the next step, each permutation will be assigned to a probability of occurrence $p(\pi)$ depending only on the number of inversions $l(\pi) = \#\{(i,j) \in \{1, 2, \ldots, n\}^2 : i < j, \pi(i) > \pi(j)\}$ and the noise parameter of an expert $\lambda \in [0, 1]$. Meaning, expert A observes permutation π with probability of occurrence $p_A(\pi)$. Bärthel defined this probability of occurrence of a permutation $\pi \in S_n$ and the noise parameter of an expert $\lambda \in [0, 1]$ in her report as

$$p_A(\pi) = \frac{\lambda^{l(\pi)}}{p_n(\lambda)}$$

with

$$\begin{aligned} p_n(\lambda) &= 1 \cdot (1 + \lambda) \cdot (1 + \lambda + \lambda^2) \cdot \ldots \cdot (1 + \lambda + \ldots + \lambda^{n-1}) \\ &= p_{n-1}(\lambda) \cdot (1 + \lambda + \ldots + \lambda^{n-1}). \end{aligned}$$

The mth coefficient of this polynomial represents the number of permutations $\pi \in S_n$ with m inversions ([Mar2001]).

π	$l(\pi)$	$p_A(\pi)$
$(1,2,3)$	0	$\frac{\lambda^0}{p_3(\lambda)} = \frac{1}{1+2\lambda+2\lambda^2+\lambda^3}$
$(1,3,2)$	1	$\frac{\lambda^1}{p_3(\lambda)} = \frac{\lambda}{1+2\lambda+2\lambda^2+\lambda^3}$
$(2,1,3)$	1	$\frac{\lambda^1}{p_3(\lambda)} = \frac{\lambda}{1+2\lambda+2\lambda^2+\lambda^3}$
$(2,3,1)$	2	$\frac{\lambda^2}{p_3(\lambda)} = \frac{\lambda^2}{1+2\lambda+2\lambda^2+\lambda^3}$
$(3,1,2)$	2	$\frac{\lambda^2}{p_3(\lambda)} = \frac{\lambda^2}{1+2\lambda+2\lambda^2+\lambda^3}$
$(3,2,1)$	3	$\frac{\lambda^3}{p_3(\lambda)} = \frac{\lambda^3}{1+2\lambda+2\lambda^2+\lambda^3}$

Table 3.4.1 Probabilities of occurrence of all permutations for $n = 3$ and noise parameter of expert A $\lambda \in [0, 1]$. Adapted from [Bae2011] with permission by Bärthel.

Obviously, the total sum of all these probabilities is 1 ($\sum_{\pi \in S_3} p_A(\pi) = 1$). Assuming $\lambda = 0$, $p_A(1,2,3) = 1$ and $p_A(\pi) = 0$ for each $\pi \in S_3 \backslash \{(1,2,3)\}$. So, expert A is perfect. And if $\lambda = 1$, $p_A(\pi) = \frac{1}{6}$ for each $\pi \in S_3$. So, expert A is totally noised.

With these agreements we now present some theoretical results Bärthel showed in her report. The probability of selecting k out of n alternatives including the best alternative 1 is defined as $P(n,k,\{1\})$.

Firstly, in Theorem 3.4.1 experts A and B are equally good with noise parameter $\lambda \in (0,1)$. Regarding this theorem scenario AB is better than scenario A.

Theorem 3.4.1 *For all $n \in \mathbb{N}$, $n \geq 3$, and for all $\lambda \in (0,1)$ is*

(i) $P_A(n,2,\{1\}) < P_{AB}(n,2,\{1\})$ *and*

(ii) $P_A(n,2,\{1,2\}) < P_{AB}(n,2,\{1,2\})$.

Now, experts A and B are not equally good. In detail, expert A with noise parameter $\lambda \in (0,1)$ is better than expert B with noise parameter $\mu \in (0,1)$. As shown in Theorem 3.4.2 scenario BA is better than scenario AB.

Theorem 3.4.2 *For all $n \in \mathbb{N}$, $n \geq 3$, and for all $\lambda, \mu \in (0,1)$ with $\lambda < \mu$ is*

(i) $P_{AB}(n,2,\{1\}) < P_{BA}(n,2,\{1\})$ *and*

(ii) $P_{AB}(n,2,\{1,2\}) < P_{BA}(n,2,\{1,2\})$.

In the last theorem presented, Theorem 3.4.3, expert A is worse than expert B. The noise parameters are as explained above. So, scenario A is worse than scenario BA and scenario BA is worse than scenario AB.

Theorem 3.4.3 *For all $n \in \mathbb{N}$, $n \geq 3$, and for all $\lambda, \mu \in (0,1)$ with $\lambda > \mu$ is*

(i) $P_A(n,2,\{1\}) < P_{BA}(n,2,\{1\}) < P_{AB}(n,2,\{1\})$ *and*

(ii) $P_A(n,2,\{1,2\}) < P_{BA}(n,2,\{1,2\}) < P_{AB}(n,2,\{1,2\})$.

To comprehend these theorems, double-expert-scenarios are better than single-expert-scenarios for equally good experts. Considering two experts with different noise parameters selecting alternately, scenarios starting with the worse expert are better than these ones starting with the better expert.

4 Results of the Selection Problem with Two Experts

After explaining the Selection Problem with two experts in Section 2.1 on page 17 this chapter contains several results of computational experiments. In addition, there are more results in Appendix A on page 87. All applied algorithms are implemented using programming language Java™. Each experiment is run with $T = 10^p$ ($p \in \{8, 9\}$) simulations and the results have p decimals. Unless otherwise specified we suppose $k \in \{0, 2, \dots, n\}$.

4.1 Experts with Equal Noise Levels

In this section we analyse the Selection Problem with two equally good experts (see Definition 2.1.1 on page 18). Therefore, we compare all seven scenarios as defined in Section 2.1. Firstly, we compare the single-expert-scenario A with the pure selection scenario AB and the pure deletion scenario ab. Figure 4.1.1 shows the performance differences between these scenarios for $k \in \{0, 2, \dots, n\}$.

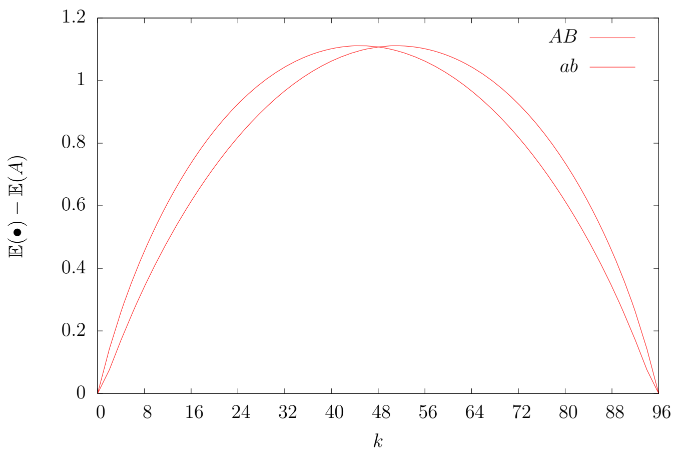


Figure 4.1.1 Scenarios AB and ab for $n = 96$, $v_a = v_b = \frac{1}{4}$, and $T = 10^8$ runs.

As illustrated both double-expert-scenarios are better than the single-expert-scenario. In this case *better* means that the expected total weight of each considered double-expert-scenario is larger than the expected total weight of scenario A. But neither AB nor ab is uniformly best for all k. Therefore, Figure 4.1.2 outlines the absolute performance differences between these scenarios for the noise levels $v_a = v_b \in \{\frac{1}{8}, \frac{1}{4}, \frac{1}{2}\}$.

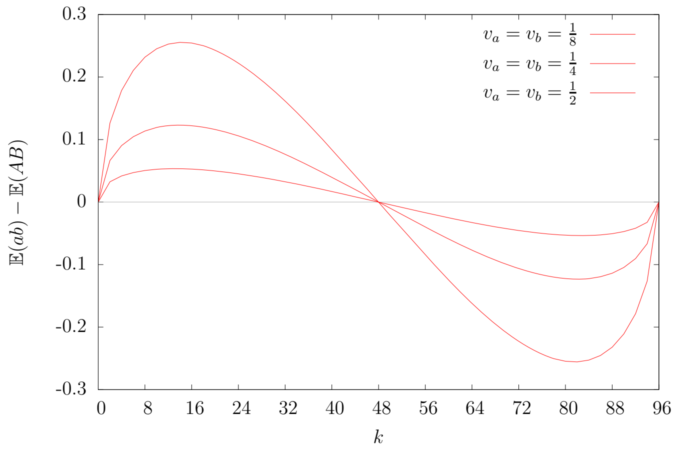


Figure 4.1.2 Absolute performance differences between the complementary scenarios ab and AB for $n = 96$, and $T = 10^8$ runs.

The experiment shows that

- ab is better than AB for $0 < k < \frac{n}{2}$,
- AB is better than ab for $\frac{n}{2} < k < n$,
- ab and AB are equal for $k \in \{0, \frac{n}{2}, n\}$, and
- the performance difference between ab and AB decreases when the expert noise levels decrease.

After these basic insights we ask two additional questions. Firstly, are the single mixed scenarios Ab and aB and the double mixed scenarios $ABab$ and $abAB$ even better than the pure selection scenario AB and the pure deletion scenario ab? And secondly, do the performance differences between the other complementary scenarios aB and Ab or $abAB$ and $ABab$ show similar characteristics as between ab and AB?

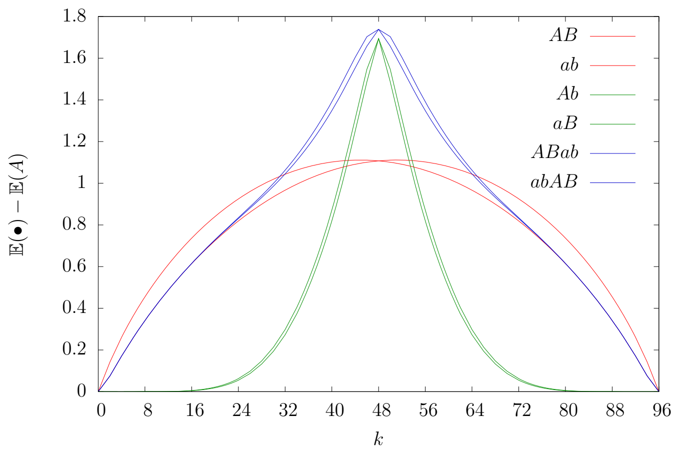


Figure 4.1.3 Scenarios AB, ab, Ab, aB, $ABab$, and $abAB$ for $n = 96$, $v_a = v_b = \frac{1}{4}$, and $T = 10^8$ runs.

Figure 4.1.3 includes six pure *and* mixed scenarios we analysed. It is obviously that we can not clearly fix the answer to the first question. In fact none of the scenarios is uniformly best for all k. But both the single mixed scenarios Ab and aB are always worse than the double mixed scenarios $ABab$ and $abAB$. So we should use a combination of ab, $abAB$, $ABab$, and AB to get the best results. In the given example use

- ab for $k \le \frac{1}{3}n$,
- $abAB$ for $\frac{1}{3}n \le k \le \frac{1}{2}n$,
- $ABab$ for $\frac{1}{2}n \le k \le \frac{2}{3}n$, and
- AB for $k \ge \frac{2}{3}n$.

The answer to the second question is much easier and presented in Figure 4.1.4. Other noise levels are figured out in Appendix A.1 on page 87.

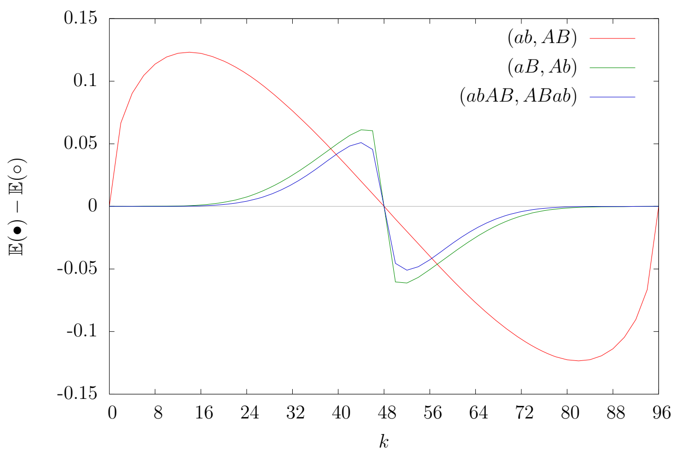


Figure 4.1.4 Absolute performance differences between pure and mixed complementary scenarios for $n = 96$, $v_a = v_b = \frac{1}{4}$, and $T = 10^8$ runs.

Among ab and AB scenarios starting with a deletion action are better than their complementary scenarios if $0 < k < \frac{n}{2}$. But whereas $\mathbb{E}(ab) - \mathbb{E}(AB)$ distinguishes most around $k = \frac{1}{8}n$ and $k = \frac{7}{8}n$, $\mathbb{E}(aB) - \mathbb{E}(Ab)$ and $\mathbb{E}(abAB) - \mathbb{E}(ABab)$ differ most in the neighbourhood of $k = \frac{n}{2}$.

One reason for the occurred effect between the complementary scenarios might be the number of *crucial* actions performed by experts A and B in these scenarios.

Definition 4.1.1 *A selection or deletion action of experts A or B is called crucial if there exist at least two possible items to select or to delete.*

Example Consider $n = 8$, $k = 6$, and the single mixed scenario aB. With the actions a, B, and a one item is already selected. But after also deleting two items, we *must* select the five remaining items. So, scenario aB stops after three crucial actions.

Table 4.1.1 clearly represents the numbers of crucial selection and deletion actions of experts A and B for $n = 96$ and $k \in \{2, 4, 30, \dots, 66, 92, 94\}$. The coloured cells highlight the maximum number of crucial actions for each considered k. How to calculate these numbers is described in Appendix A.1 (see Table A.1.1 on page 88).

k	**AB**		**ab**		**Ab**		**aB**		**ABab**				**abAB**			
	A	B	a	b	A	b	a	B	A	B	a	b	a	b	A	B
2	1	1	47	47	2	1	2	2	1	1	0	0	1	1	1	1
4	2	2	46	46	4	3	4	4	2	2	1	1	2	2	2	2
⋮																
30	15	15	33	33	30	29	30	30	15	15	14	14	15	15	15	15
32	16	16	32	32	32	31	32	32	16	16	15	15	16	16	16	16
34	17	17	31	31	34	33	34	34	17	17	16	16	17	17	17	17
36	18	18	30	30	36	35	36	36	18	18	17	17	18	18	18	18
38	19	19	29	29	38	37	38	38	19	19	18	18	19	19	19	19
40	20	20	28	28	40	39	40	40	20	20	19	19	20	20	20	20
42	21	21	27	27	42	41	42	42	21	21	20	20	21	21	21	21
44	22	22	26	26	44	43	44	44	22	22	21	21	22	22	22	22
46	23	23	25	25	46	45	46	46	23	23	22	22	23	23	23	23
48	24	24	24	24	48	47	48	47	24	24	23	23	24	24	23	23
50	25	25	23	23	46	46	46	45	23	23	23	23	23	23	22	22
52	26	26	22	22	44	44	44	43	22	22	22	22	22	22	21	21
54	27	27	21	21	42	42	42	41	21	21	21	21	21	21	20	20
56	28	28	20	20	40	40	40	39	20	20	20	20	20	20	19	19
58	29	29	19	19	38	38	38	37	19	19	19	19	19	19	18	18
60	30	30	18	18	36	36	36	35	18	18	18	18	18	18	17	17
62	31	31	17	17	34	34	34	33	17	17	17	17	17	17	16	16
64	32	32	16	16	32	32	32	31	16	16	16	16	16	16	15	15
66	33	33	15	15	30	30	30	29	15	15	15	15	15	15	14	14
⋮																
92	46	46	2	2	4	4	4	3	2	2	2	2	2	2	1	1
94	47	47	1	1	2	2	2	1	1	1	1	1	1	1	0	0

Table 4.1.1 Number of crucial selection and deletion actions of A and B in the complementary scenarios for $n = 96$ and $k \in \{2, 4, 30, \dots, 66, 92, 94\}$.

As mentioned above, scenarios starting with a deletion action are better than their complementary scenarios if $0 < k < \frac{n}{2}$. And if $\frac{n}{2} < k < n$, this effect is reverse. To give an example, for $k = 40$ the numbers of crucial actions in the scenarios *ab*, *aB*, and *abAB* are larger than in their complementary scenarios. For $k = 66$, scenarios starting with a selection action are better than their complementary scenario. So, within a pair of complementary scenarios the scenario with larger number of crucial actions is better. It seems natural that larger numbers of crucial actions lead to better performances in general. But this conclusion is not true since *ABab* and *abAB* are always better than *Ab* and *aB* (see Figure 4.1.3 on page 36) even though the number of crucial actions is equal or less.

After having examined the results of the pure and mixed scenarios we will now explicitly deal with pure selection scenarios. Until now, we declared one fixed selection order (scenario AB) where both experts act in equal parts ($\#A = \#B = \frac{k}{2}$). In order to improve the results we now allow $\#A \neq \#B$ and consider all permutations of these selection orders. The number of permuted selection orders depending on the selection number k is given by

$$r(k) := \begin{cases} \binom{k}{\frac{k}{2}} & \text{if } \#A = \#B = \frac{k}{2} \text{ and } k \text{ is even,} \\ \\ 2^k & \text{if } \#A + \#B = k. \end{cases}$$

Due to $v_a = v_b$ we only consider A-starting scenarios for $\#A = \#B$ and scenarios with $\#A > \#B$. Under this restriction the number of permuted selection orders is $\frac{1}{2} \cdot r(k)$.

To get a first impression how variable selection orders can improve the result, Table 4.1.2 shows the ranking of the pure selection scenarios for $n = 96$, $k = 4$, and $v_a = v_b = \frac{1}{4}$. So, there are $\frac{1}{2} \cdot 2^4 = 8$ of them in total. In the following, a pure selection scenario with i A-actions and j B-actions is called $A^i B^j$-scenario with $i + j = k$ and $i \geq j$. In the current example there are 3 A^2B^2-scenarios, 4 A^3B^1-scenarios, and one A^4B^0-scenario.

rank	#A	#B	selection order				$\sum_{i \in S_4} x_i$
1	2	2	A	B	B	A	7.6290
2	2	2	**A**	**B**	**A**	**B**	7.6269
3	2	2	A	A	B	B	7.6185
4	3	1	A	A	A	B	7.6118
5	3	1	A	A	B	A	7.6001
6	3	1	A	B	A	A	7.5815
7	3	1	B	A	A	A	7.5514
8	4	0	A	A	A	A	7.4564

Table 4.1.2 Ranking of pure selection scenarios for $n = 96$, $k = 4$, $v_a = v_b = \frac{1}{4}$, and $T = 10^9$ runs (average values rounded to four decimals).

As illustrated, the best three results are the ones where experts A and B act in equal parts ($\#A = \#B = 2$). Within these three A^2B^2-scenarios $ABBA$ is the best and $AABB$ is the worst. So, permutations of scenario AB can lead to better **or** worse results. The next block of rankings includes the A^3B^1-scenarios and shows an interesting structure. Allowing expert B to select only once, he should act **as late as possible** (scenario $AAAB$). Otherwise, the average total sum of the selected items becomes smaller. The result is worst if expert B selects the first item and expert A selects the other three items (scenario $BAAA$). The last row of Table 4.1.2 includes the result of scenario A. As figured out earlier in this chapter, scenario AB is better than scenario A for the currently considered parameters (see Figure 4.1.1 on page 35). But as shown in the previous table each scenario with at least one B-action is better than scenario A.

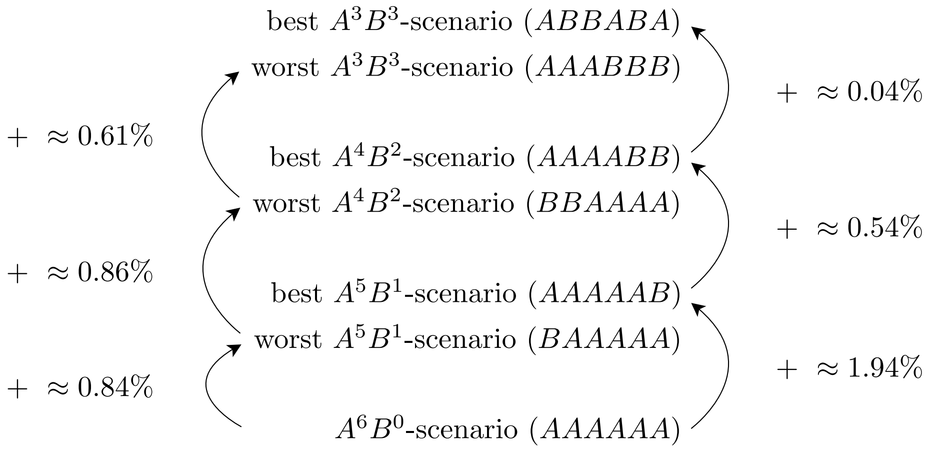
For $n = 96$ and $k = 4$ the A^2B^2- and A^3B^1-scenarios form blocks (see Table 4.1.2 on page 39). But this is not the general case for other parameters n and k. Table 4.1.3 gives an overview how all A^3B^3-, A^4B^2, A^5B^1 and A^6B^0-scenarios ($k = 6$) are ranked for $n \in \{12, 24, 48, 96, 192\}$ and $v_a = v_b = \frac{1}{4}$. There are rankings including specific selection orders for these parameters presented in Appendix A.1 starting on page 90.

rank	$n = 12$	$n = 24$	$n = 48$	$n = 96$	$n = 192$	$n = 384$
1	A^3B^3	A^3B^3	A^3B^3	A^3B^3	A^3B^3	A^3B^3
2	A^4B^2	A^3B^3	A^3B^3	A^3B^3	A^3B^3	A^3B^3
3	A^3B^3	A^3B^3	A^3B^3	A^3B^3	A^3B^3	A^3B^3
4	A^4B^2	A^4B^2	////////	////////	A^3B^3	A^3B^3
5	A^3B^3	A^3B^3	A^4B^2	A^3B^3	////////	////////
6	A^3B^3	A^4B^2	A^4B^2	A^3B^3	A^3B^3	A^3B^3
7	A^4B^2	////////	A^3B^3	A^4B^2	A^3B^3	A^3B^3
8	A^4B^2	A^4B^2	A^3B^3	A^4B^2	A^3B^3	A^3B^3
9	////////	A^4B^2	A^4B^2	A^3B^3	A^4B^2	A^3B^3
10	A^4B^2	A^3B^3	A^4B^2	A^3B^3	A^4B^2	A^4B^2
11	A^4B^2	A^4B^2	A^3B^3	A^4B^2	A^3B^3	A^4B^2
12	A^4B^2	A^3B^3	A^3B^3	A^4B^2	A^4B^2	A^3B^3
13	A^3B^3	A^4B^2	A^4B^2	A^3B^3	A^4B^2	A^4B^2
14	A^5B^1	A^3B^3	A^4B^2	A^4B^2	A^3B^3	A^4B^2
15	A^3B^3	A^4B^2	A^3B^3	A^4B^2	A^4B^2	A^4B^2
16	A^4B^2	A^4B^2	A^4B^2	A^3B^3	A^4B^2	A^4B^2
17	A^3B^3	A^3B^3	A^4B^2	A^4B^2	A^4B^2	A^4B^2
18	A^4B^2	A^4B^2	A^4B^2	A^4B^2	A^4B^2	A^4B^2
19	A^4B^2	A^5B^1	A^3B^3	A^4B^2	A^4B^2	A^4B^2
20	A^3B^3	A^4B^2	A^4B^2	A^4B^2	A^4B^2	A^4B^2
21	A^5B^1	A^4B^2	A^4B^2	A^4B^2	A^4B^2	A^4B^2
22	A^4B^2	A^3B^3	A^5B^1	A^4B^2	A^4B^2	A^4B^2
23	A^4B^2	A^5B^1	A^4B^2	A^4B^2	A^4B^2	A^4B^2
24	A^3B^3	A^4B^2	A^4B^2	A^5B^1	A^4B^2	A^4B^2
25	A^4B^2	A^4B^2	A^5B^1	A^4B^2	A^5B^1	A^4B^2
26	A^5B^1	A^4B^2	A^4B^2	A^5B^1	A^4B^2	A^5B^1
27	A^4B^2	A^5B^1	A^5B^1	A^4B^2	A^5B^1	A^5B^1
28	A^5B^1	A^4B^2	A^4B^2	A^5B^1	A^5B^1	A^5B^1
29	A^4B^2	A^5B^1	A^5B^1	A^5B^1	A^5B^1	A^5B^1
30	A^5B^1	A^5B^1	A^5B^1	A^5B^1	A^5B^1	A^5B^1
31	A^5B^1	A^5B^1	A^5B^1	A^5B^1	A^5B^1	A^5B^1
32	A^6B^0	A^6B^0	A^6B^0	A^6B^0	A^6B^0	A^6B^0

A^3B^3 A^4B^2 A^5B^1 A^6B^0

Table 4.1.3 Ranking of A^iB^j-scenarios for $n \in \{12, 24, 48, 96, 192, 384\}$, $k = 6$, $v_a = v_b = \frac{1}{4}$, and $T = 10^9$ runs. Hatched cells mean a strictly alternating selection order ($ABABAB$).

As mentioned above, the different A^iB^j-scenarios do not form blocks any more. That means not all of these scenarios where expert A and expert B act in equal parts are better than scenarios with $\#A > \#B$. The best A^4B^2-scenarios lead to good or very good results as well. Even the best A^5B^1-scenario ($AAAAAB$) is better than up to four out of ten A^3B^3-scenarios and is located in the upper half of the ranking ($n = 12$). By increasing the number of the items n the block structure increasingly comes out. So, for $n = 384$ there is just one exception ($AAABBB$). As already illustrated in Figure 4.1.1 on page 35, scenario AB is better than scenario A. Here all scenarios with at least one action of the equally good expert B are better than the A^6B^0-scenario. This observation holds for all considered $k \in \{3,4,5,6\}$ (see Appendix A.1 starting on page 87). The level of the improvement of the result caused by the employment of expert B depends on the number of B-actions. For this purpose



Comparing the best A^iB^j-scenarios, the relative improvement with a singular B-action is larger than the improvement at the transition from one to two and from two to three B-actions. So, the relative improvement decreases by increasing the number of B-actions. This illustration also shows that $AAAAAB$ is the best A^5B^1-scenario and AB is *not* the best A^3B^3-scenario. The ranking of the A^5B^1-scenarios is $AAAAAB$, $AAAABA$, ..., $BAAAAA$. So, to obtain the best result the singular employment of expert B should happen as late as possible. This observation confirms for all considered n and k. The best A^3B^3-scenarios for $v_a = v_b = \frac{1}{4}$ are $ABABBA$ for $n \in \{12, 24, 48\}$ and $ABBABA$ for $n \in \{96, 192, 384\}$.

The rankings of A^iB^j-scenarios for different noise levels also have similar characteristics. There is an overview for $n = 96$, $k = 6$, and $v_a = v_b \in \{\frac{1}{8}, \frac{1}{4}, \frac{1}{2}, 1\}$ shown in Table A.1.13 on page 95.

Now we are interested in the position of scenario AB within the $A^{\frac{k}{2}}B^{\frac{k}{2}}$-scenarios (only even k) and within all A^iB^j-scenarios for $i + j = k$ and $i \geq j$. Table 4.1.4 includes the ranking of the pure selection scenarios according to n and k. To be exact, the values in the left table represent the position of scenario AB if $\#A + \#B = k$ and the values in the right table describe the position of AB if $\#A = \#B = \frac{k}{2}$. There are detailed rankings for $n = 96$, $k \in \{3,4,5\}$ and $v_a = v_b \in \{\frac{1}{4}, \frac{1}{2}\}$ in Appendix A.1 starting on page 89.

k	$\frac{1}{2} \cdot r(k)$	n			
		12	24	48	96
3	4	2	2	2	2
4	8	3	2	2	2
5	16	5	5	5	5
6	32	9	7	4	4
7	64	18	17	14	15
8	128	36	26	21	14
9	256	74	59	55	51

k	$\frac{1}{2} \cdot r(k)$	n			
		12	24	48	96
4	3	2	2	2	2
6	10	5	5	4	4
8	35	15	14	13	12

Table 4.1.4 AB ranking for $n \in \{12,24,48,96\}$, $k \in \{3,4,\dots,9\}$, $v_a = v_b = \frac{1}{4}$, and $T = 10^9$ runs. Left: $r(k) = 2^k$ for $\#A + \#B = k$. Right: $r(k) = \binom{k}{\frac{k}{2}}$ for $\#A = \#B = \frac{k}{2}$.

As presented, AB is not the best scenario within the $A^{\frac{k}{2}}B^{\frac{k}{2}}$-scenarios and consequently not the best scenario within all possible permutations ($= \frac{1}{2} \cdot 2^k$). There is at least one (usually more than one) better selection order. If $k = 4$ and $\#A = \#B$, selection order $ABBA$ is always better. Furthermore, the position of AB becomes better by increasing the number of items and stays almost constant for large n (see $k = 6$ in Table 4.1.3 on page 40 and $k = 7$ in the table above).

4.2 Experts with Different Noise Levels

This section contains several results of the Selection Problem with two experts with different noise levels. In all experiments, the noise level of expert A is smaller than the noise level of expert B ($v_a < v_b$). So, expert A is always better than expert B. To extend Section 4.1 on page 35 we now consider scenarios starting with an action of expert B as well. These are the following seven scenarios: BA, ba, Ba, bA, $BAba$, and $baBA$. Scenario B is not listed because it is always worse than scenario A. Under these settings we are now interested in two independent key questions:

- Are scenarios starting with an action of expert B better than those starting with an action of expert A?
- Can the employment of one additional, worse expert B improve the result of the single-expert-scenario A?

Answering these two questions, Figure 4.2.1 shows the absolute performance differences between scenario A and the double-expert-scenarios AB, BA, ab, and ba.

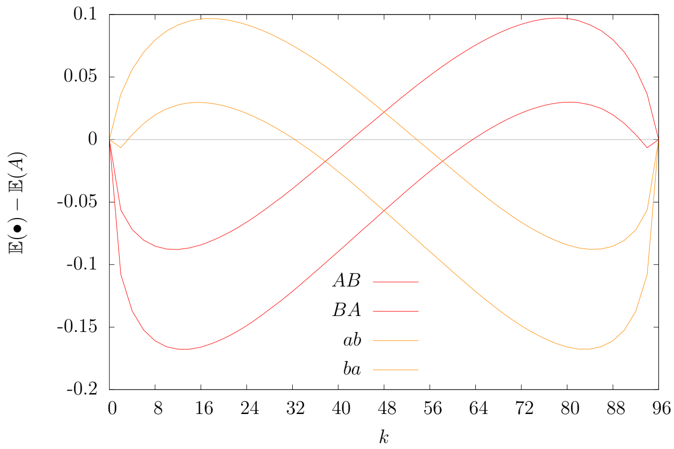


Figure 4.2.1 Scenarios AB, BA, ab, and ba for $n = 96$, $v_a = \frac{1}{4}$, $v_b = \frac{1}{2}$, and $T = 10^8$ runs.

Both B-starting scenarios BA and ba are better than their reverse scenarios AB and ab for all considered **even** k. And every plotted double-expert-scenario improves the result of the single-expert-scenario A in a particular subset of $\{0, 2, \ldots, 96\}$ even though expert B is worse than expert A. With a combination of scenarios ba ($k \in \{0, 2, \ldots, 48\}$) and BA ($k \in \{48, 50, \ldots, 96\}$) we totally improve the result of scenario A. BA is better than AB and ba is better than ab for all $v_b \in \{\frac{3}{8}, \frac{5}{8}, \frac{3}{4}, \frac{7}{8}, 1\}$, too. Figure 4.2.2 shows how both B-starting scenarios BA and ba for the noise levels mentioned perform in comparison to the result of the single-expert-scenario A.

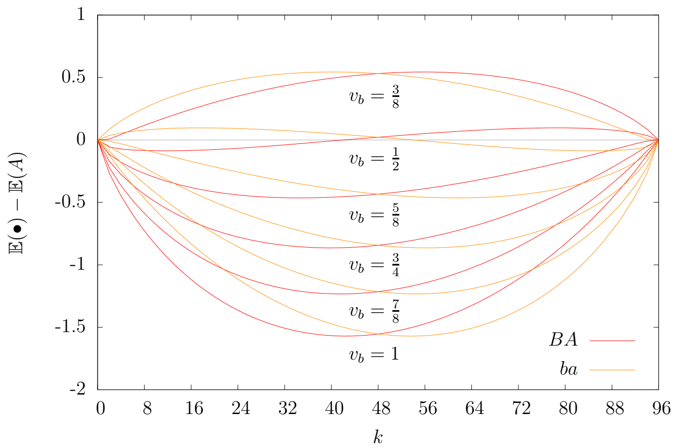


Figure 4.2.2 Scenarios BA and ba for $n = 96$, $v_a = \frac{1}{4}$, $v_b \in \{\frac{3}{8}, \frac{5}{8}, \frac{3}{4}, \frac{7}{8}, 1\}$, and $T = 10^8$ runs.

Whereas the result of scenario A can be improved for $v_b \in \{\frac{3}{8}, \frac{1}{2}\}$ and with the combination of ba and BA as described previously, the employment of expert B with noise level $v_b \in \{\frac{5}{8}, \frac{3}{4}, \frac{7}{8}, 1\}$ leads to a worse result.

Beside these B-starting pure selection and pure deletion scenarios, also the double mixed scenarios $BAba$ and $baBA$ are better than their reverse scenarios $Abab$ and $abAB$. For $v_a = \frac{1}{4}$ and $v_b = \frac{1}{2}$ the results of these four scenarios are plotted below in Figure 4.2.3.

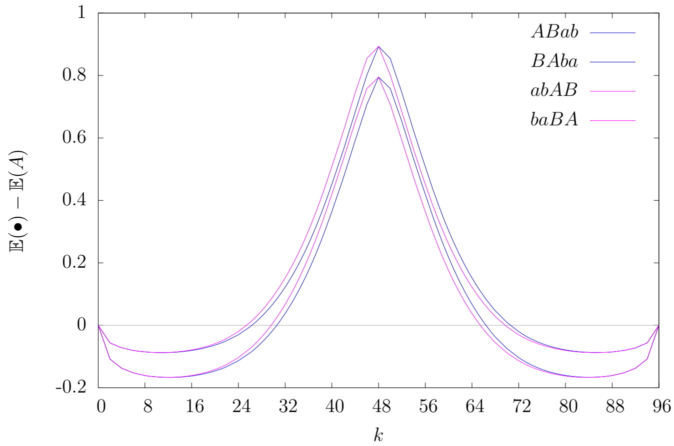


Figure 4.2.3 Scenarios $ABab$, $BAba$, $abAB$, and $baBA$ for $n = 96$, $v_a = \frac{1}{4}$, $v_b = \frac{1}{2}$, and $T = 10^8$ runs.

Here, the results of $BAba$ and $baBA$ are also better than the result of scenario A for $k \in \{26, 28, \ldots, 70\}$. Now we consider the four remaining single mixed scenarios Ab, Ba, aB, and bA for $v_a = \frac{1}{4}$ and $v_b = \frac{1}{2}$, too. The results are presented in Figure 4.2.4.

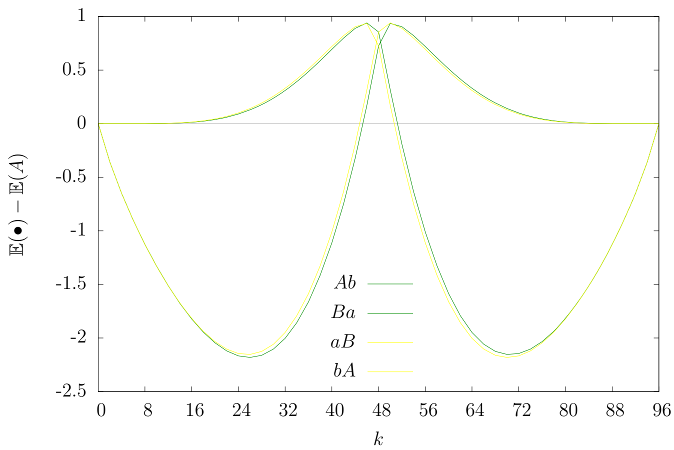


Figure 4.2.4 Scenarios Ab, Ba, aB, and bA for $n = 96$, $v_a = \frac{1}{4}$, $v_b = \frac{1}{2}$, and $T = 10^8$ runs.

In contrast to the other four scenario pairs neither Ba nor bA are better than their reverse scenarios Ab and aB. In detail, none of these scenarios is better than at least one of the other ones. But as described previously for the B-starting scenarios BA and ba, a combination of Ab or bA ($k \in \{0, 2, \ldots, 48\}$) and Ba or bA

($k \in \{48, 50, \dots, 96\}$) totally improves the result of scenario A, too.

After studying twelve scenarios or six scenario pairs there is one more question: Which (B-starting) scenario is best? To give an answer, all considered B-starting scenarios for $v_a = \frac{1}{4}$ and $v_b = \frac{1}{2}$ are plotted in Figure 4.2.5. There is an analogous plot for all considered scenarios with equally good experts ($v_a = v_b = \frac{1}{4}$) given in Figure 4.1.3 on page 36.

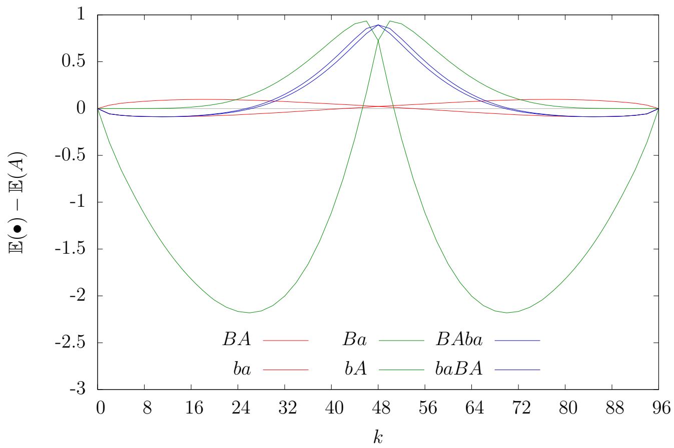


Figure 4.2.5 Scenarios BA, ba, Ba, bA, $BAba$, and $baBA$ for $n = 96$, $v_a = \frac{1}{4}$, $v_b = \frac{1}{2}$, and $T = 10^8$ runs.

Obviously, none of these scenarios is uniformly best for all considered k. Each scenario is the best one in a particular subset of $\{0, 2, \dots, 96\}$. The best scenarios are

- ab for $k \in \{2, 4, \dots, 22\}$,
- bA for $k \in \{26, 28, \dots, 46\}$,
- $BAba$ and $baBA$ for $k = 48$,
- Ba for $k \in \{50, 52, \dots, 70\}$, and
- BA for $k \in \{74, 76, \dots, 94\}$.

For $k \in \{0, \frac{n}{4}, \frac{n}{2}, \frac{3n}{4}, n\} = \{0, 24, 48, 72, 96\}$ at least two scenarios are approximately equal.

In Section 4.1 starting on page 39 we reflected upon pure selection scenarios with variable selection orders. Therefore, we defined the number of permuted selection orders depending on the selection number k as $r(k)$. Due to $v_a < v_b$ we now consider A- *and* B-starting scenarios with $\#A + \#B = k$. The $A^i B^j$-notation will be used as well. Table 4.2.1 shows the ranking of the pure selection scenarios for $n = 96$, $k = 4$, $v_a = \frac{1}{4}$, and $v_b = \frac{1}{2}$. Now there are 6 A^2B^2-scenarios, 4 A^3B^1-scenarios, 4 A^1B^3-scenarios, one A^4B^0-scenario, and one A^0B^4-scenario.

rank	#A	#B	selection order				$\sum_{i \in S_4} x_i$
1	3	1	B	A	A	A	7.4785
2	3	1	A	B	A	A	7.4687
3	4	0	**A**	**A**	**A**	**A**	7.4564
4	3	1	A	A	B	A	7.4546
5	3	1	A	A	A	B	7.4370
6	2	2	B	B	A	A	7.4093
7	2	2	**B**	**A**	**B**	**A**	7.3843
8	2	2	B	A	A	B	7.3582
9	2	2	A	B	B	A	7.3483
10	2	2	**A**	**B**	**A**	**B**	7.3192
11	2	2	A	A	B	B	7.2815
12	1	3	B	B	B	A	7.2146
13	1	3	B	B	A	B	7.1728
14	1	3	B	A	B	B	7.1233
15	1	3	A	B	B	B	7.0559
16	0	4	B	B	B	B	6.8067

Table 4.2.1 Ranking of pure selection scenarios for $n = 96$, $k = 4$, $v_a = \frac{1}{4}$, $v_b = \frac{1}{2}$, and $T = 10^9$ runs (average values rounded to four decimals).

Here, the A^2B^2- and A^1B^3-scenarios form blocks, too. Within the A^2B^2-scenarios $BBAA$ is the best one and $AABB$ the worst one. Moreover, scenario BA (rank 7) is better than scenario AB (rank 10). Altogether, the B-starting A^2B^2-scenarios are better than there complementary A-starting A^2B^2-scenarios. To achieve the best result within the A^1B^3-scenarios, expert A should act **as late as possible**. So, $BBBA$ is the best scenario in this group. The five best scenarios are these ones with at least three actions of expert A. Within the A^3B^1-scenarios expert B should act **as early as possible**. Even though expert B is worse than expert A, scenario A is worse than two scenarios with one B-action each. As shown in Table A.2.4 on page 96, increasing the noise level of expert B to $v_b = \frac{3}{4}$ leads to A as the best selection scenario. In addition to these results, there are A^iB^j-rankings for $n = 96$, $k \in \{3, 5, 6\}$, $v_a = \frac{1}{4}$, and $v_b \in \{\frac{1}{2}, \frac{3}{4}\}$ in Appendix A.2 starting on page 96.

To comprehend this section, double-expert-scenarios **can** improve the result of the single-expert-scenario although B is worse than A. Further, within the pure selection, pure deletion and double mixed scenarios each B-starting scenario is better than its reverse A-starting scenario.

5 Results of the Selection Problem with k Experts

This chapter discusses selected results of k-SeP described in Section 2.2 on page 20. Additional and more detailed results are shown in Appendix B starting on page 101. All applied algorithms are implemented using programming language Java™. T describes the number of simulations of an experiment.

In our experiments we used two different distributions for the noise levels of the experts. These are defined as follows.

(1) $v_1, v_2, \ldots, v_k$ equidistant in $\left[\frac{1}{k}, 1\right]$.

(2) $v_1, v_2, \ldots, v_{k-1}$ equidistant in $\left[\frac{1}{2(k-1)}, \frac{1}{2}\right]$ and $v_k = 1$.

As described in Model 2.2.1 on page 20, we consider $v_1 < v_2 < \ldots < v_k$.

Example Consider $k = 4$. Then

$$v^{(1)} = \left(\frac{1}{4}, \frac{2}{4}, \frac{3}{4}, \frac{4}{4}\right) = \left(\frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1\right)$$

and

$$v^{(2)} = \left(\frac{1}{6}, \frac{2}{6}, \frac{3}{6}, \frac{6}{6}\right) = \left(\frac{1}{6}, \frac{1}{3}, \frac{1}{2}, 1\right).$$

In our experiments we considered $k \in \{3, 4, 5, 6\}$ experts and $n \in \{k+1, 2k, 3k, 4k\}$ items. An extensive overview of the results of these experiments is included in Appendix B starting on page 101.

It becomes apparent that the best selection order is independent of the number of the considered experts, the number of items, and the noise level distribution. The best selection order is

$$k \quad k-1 \quad k-2 \quad k-3 \quad \ldots \quad 1.$$

So, the worst expert selects the first item, the next best expert selects the second item and so on until the best expert selects the kth item. In contrast to the best selection order, the next best selection orders with (1) depend on n and those with (2) depend on n and k. For both noise level distributions the order of these selection orders becomes stable by increasing n. So, for $n = 4k$ and (1) the second and third best selection orders are

$$k-1 \quad k \quad k-2 \quad k-3 \quad \ldots \quad 1 \quad \text{and}$$
$$k \quad k-2 \quad k-1 \quad k-3 \quad \ldots \quad 1.$$

As mentioned above, the second and third best selection orders with (2) depend on n and k. Considering $k \in \{5, 6\}$, $n = 4k$, and (2), the second and third best selection orders are

$$\begin{array}{cccccc} k & k-2 & k-1 & k-3 & \dots & 1 \quad \text{and} \\ k & k-1 & k-3 & k-2 & \dots & 1. \end{array}$$

For $k = 3$ and $n = 12$ the second and third best selection orders with (2) are equal to those with (1). And for $k = 4$ and $n = 16$ the second (third) best selection order with (2) is equal to the third (second) best selection order with (1).

In contrast to the best selection orders, the (five) worst selection orders with (1) depend also on the number of considered experts. Therefore Table 5.1 represents the ranking of the five worst selections orders of this distribution for $k \in \{3, 4, 5, 6\}$ and $n \in \{k + 1, 2k, 3k, 4k\}$. Framed pairs of experts symbolise neighbouring 2-exchanges compared with the worst selection order for each k and n. For these parameters there are entire rankings shown in Tables B.1, B.3, B.5, and B.7 starting on page 102.

k	$n = k + 1$	$n = 2k$	$n = 3k$	$n = 4k$
	2 3 1	2 3 1	2 3 1	2 3 1
	3 1 2	3 1 2	3 1 2	3 1 2
3	1 3 2	2 1 3	2 1 3	2 1 3
	2 1 3	1 3 2	1 3 2	1 3 2
	1 2 3	1 2 3	1 2 3	1 2 3
	3 1 2 4	2 1 4 3	2 1 4 3	2 1 4 3
	1 2 4 3	1 3 2 4	2 1 3 4	2 1 3 4
4	1 3 2 4	1 2 4 3	1 3 2 4	1 3 2 4
	2 1 3 4	2 1 3 4	1 2 4 3	1 2 4 3
	1 2 3 4	1 2 3 4	1 2 3 4	1 2 3 4
	3 1 2 4 5	1 2 3 5 4	1 3 2 4 5	2 1 3 4 5
	1 2 4 3 5	1 2 4 3 5	2 1 3 4 5	1 3 2 4 5
5	1 3 2 4 5	1 3 2 4 5	1 2 4 3 5	1 2 4 3 5
	2 1 3 4 5	2 1 3 4 5	1 2 3 5 4	1 2 3 5 4
	1 2 3 4 5	1 2 3 4 5	1 2 3 4 5	1 2 3 4 5
	2 3 1 4 5 6	1 2 4 3 5 6	1 2 4 3 5 6	2 1 3 4 5 6
	3 1 2 4 5 6	1 2 3 5 4 6	1 2 3 5 4 6	1 2 4 3 5 6
6	1 3 2 4 5 6	1 3 2 4 5 6	1 2 3 4 6 5	1 2 3 5 4 6
	2 1 3 4 5 6	2 1 3 4 5 6	2 1 3 4 5 6	1 2 3 4 6 5
	1 2 3 4 5 6	1 2 3 4 5 6	1 2 3 4 5 6	1 2 3 4 5 6

Table 5.1 Ranking of the five worst selection orders for $k \in \{3, 4, 5, 6\}$, $n \in \{k + 1, 2k, 3k, 4k\}$, $v = (v_1, v_2, \dots, v_k)$ equidistant in $\left[\frac{1}{k}, 1\right]$, and $T = 10^9$ runs.

Obviously, the worst selection order is independent of k and n and is

$$1 \quad 2 \quad \ldots \quad \text{k}.$$

While the next best selection orders behave similarly for different k, they differ depending on the number of items n. So especially for $n = k + 1$ the ranking is quite mixed. By increasing n to $4k$ the structure of these selection orders becomes stable. Such behaviour was also observed for the best selection orders (see previous page). So, we get the ith worst selection order by reversing the position of expert $k - i + 1$ and expert $k - i + 2$ for $1 < i \leq k$. There is an exceptional case for $k = 6$ where the fifth and sixth worst selection orders are swapped (see Table B.7 on page 107). After reversing one pair of experts we get the next best selection orders by reversing two or more pairs of experts (see $k \in \{3, 4\}$), until we get the best selection order described on page 47.

In the following, we have a look at the differences between the results with (1) and (2). For this purpose Table 5.2 includes the five best and the five worst selection orders for both noise level distributions, $k = 6$, and $n = 24$. Table B.7 and Table B.8 on pages 107 and 108 represent an extensive ranking for these parameters.

rank	noise level distribution (1)	noise level distribution (2)
1	6 5 4 3 2 1	6 5 4 3 2 1
2	[5 6] 4 3 2 1	6 [4 5] 3 2 1
3	6 [4 5] 3 2 1	6 5 [3 4] 2 1
4	6 5 [3 4] 2 1	6 5 4 [2 3] 1
5	[5 6] [3 4] 2 1	[5 6] 4 3 2 1
716	[2 1] 3 4 5 6	1 2 [4 3] 5 6
717	1 2 [4 3] 5 6	1 [3 2] 4 5 6
718	1 2 3 [5 4] 6	1 2 3 [5 4] 6
719	1 2 3 4 [6 5]	[2 1] 3 4 5 6
720	1 2 3 4 5 6	1 2 3 4 5 6

Table 5.2 Ranking of the five best and the five worst selection orders for $k = 6$, $n = 24$, $v^{(1)} = \left(\frac{1}{6}, \frac{2}{6}, \frac{3}{6}, \frac{4}{6}, \frac{5}{6}, \frac{6}{6}\right)$, $v^{(2)} = \left(\frac{1}{10}, \frac{2}{10}, \frac{3}{10}, \frac{4}{10}, \frac{5}{10}, \frac{10}{10}\right)$, and $T = 10^9$ runs.

Obviously, the result of a selection order with (2) is better than the same one with (1) regarding the fact expert i with (2) is at least as good as expert i with (1) ($v_i^{(2)} \leq v_i^{(1)}$ for $i = 1, 2, \ldots, 6$).

While $5 \leftrightarrow 6$ (reversing the position of expert 5 and expert 6) with (1) obtains the smallest deviation from the best selection order 6 5 4 3 2 1, $5 \leftrightarrow 6$ with (2) leads to a clearly impairment of the ranking (four positions lost). The reason is because $v_5^{(2)} \ll v_6^{(2)}$. In contrast, we achieve better results with singular, neighbouring 2-exchanges of experts 2, 3, 4, and 5. With (1) it also turns out that two neighbouring

2-exchanges (5 ↔ 6 and 3 ↔ 4) lead to a better result than one neighbouring 2-exchange of the three best experts 1, 2, and 3 (1 ↔ 2 or 2 ↔ 3) does. Changing the position of expert 1 comes to a large impairment of the best selection order with both noise level distributions. With (1), $k = 6$, and $n = 24$ the best selection order, where expert 1 does not perform the sixth selection action, is on rank 19 (6 5 4 3 1 2). With (2) it is on rank 13 (also 6 5 4 3 1 2).

Furthermore, we look at the position of expert 1 within the (five) worst selection orders. Here, we obtain a larger improvement for 1 ↔ 2 with (1) as for this reversal with (2), because $v_2^{(1)} - v_1^{(1)} > v_2^{(2)} - v_1^{(2)}$. As already mentioned, expert 6 with (2) is clearly worse than expert 5 with (2), meaning $v_5^{(2)} \ll v_6^{(2)}$. Consequently for the worst selection orders, 5 ↔ 6 with (1) leads to the smallest improvement compared to the worst selection order 1 2 3 4 5 6. With (2) the improvement is explicitly larger. So, 1 2 3 4 6 5 (best selection order for 5 ↔ 6) is positioned on the 704th out of 720 ranks. Similarly to the five best selection orders with (1), here selection orders with two or more neighbouring 2-exchanges are ranked behind.

6 Results of the Team Selection Problem

In this chapter we analyse the Team Selection Problem (TeSeP) as explained in Section 2.3 starting on page 20. We consider three experts with equal or different noise levels. These experts compete per team against each other (experts A and B forms a team against expert C). There are additional results of TeSeP included in Appendix C starting on page 109. All applied algorithms are implemented using programming language Java™. T describes the number of simulations of an experiment. Unless otherwise specified we suppose $k \in \{0, 2, \ldots, \frac{n}{2}\}$.

In the analysis of the TeSeP-results we distinguish four different cases of the noise levels.

(1) $v_a = v_b = v_c$ (A, B, and C are equally good.)

(2) $v_a = v_b > v_c$ (A and B are equally good and C is better than A and B.)

(3) $v_a = v_c < v_b$ (A and C are equally good and B is worse than A and C.)

(4) $v_a < v_c < v_b$ (A is better than C and C is better than B.)

If $v_a = v_b$ (cases (1) and (2)), we consider scenarios $AB : C$ (selection order $ACBC$) and $C : AB$ (selection order $CACB$). So, in the AB-team expert A always starts.

As shown in Section 4.1 starting on page 35, scenario AB (and scenario ab) is better than scenario A. We now ask: Does team AB win if $v_a = v_b = v_c$ (case (1))? Therefore Figure 6.1 includes the performance differences between team AB and team C for $n = 96$ and $v_a = v_b = v_c = \frac{1}{4}$.

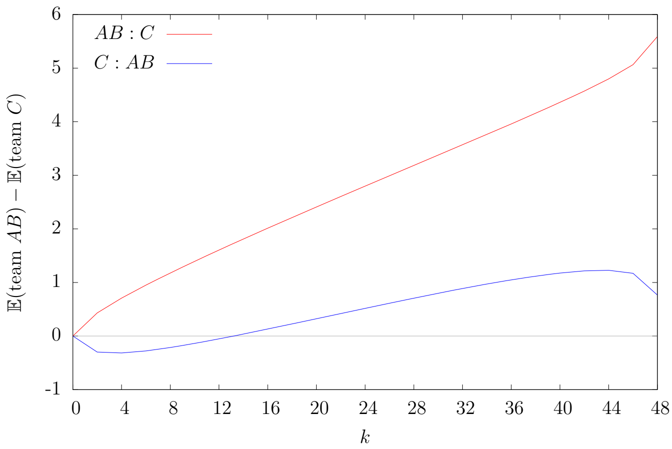


Figure 6.1 Scenarios $AB : C$ and $C : AB$ for $n = 96$, $v_a = v_b = v_c = \frac{1}{4}$, and $T = 10^8$ runs.

Considering scenario $AB : C$ (team AB always starts selecting), team AB always wins. But if C always starts selecting, he defeats AB for $k \in \{2, 4, 6, 8, 10, 12\}$ items to select. Answering the question whether the cardinality of this set of selection numbers increases or decreases by increasing the noise levels, Figure 6.2 shows scenario $C : AB$ for $n = 96$ and $v_a = v_b = v_c \in \{\frac{1}{8}, \frac{1}{4}, \frac{1}{2}, 1\}$.

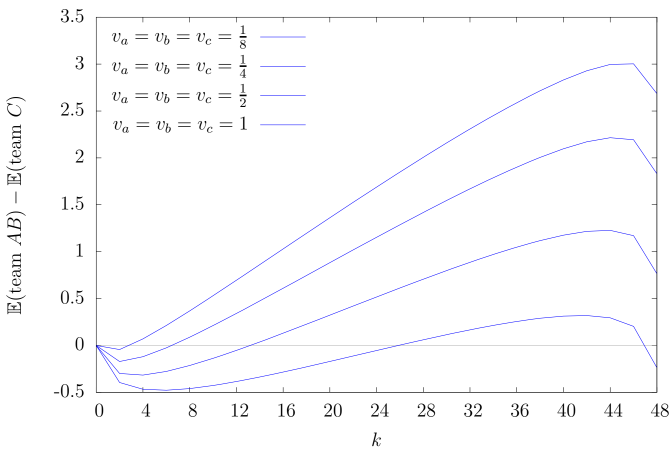


Figure 6.2 Scenario $C : AB$ for $n = 96$, $v_a = v_b = v_c \in \{\frac{1}{8}, \frac{1}{4}, \frac{1}{2}, 1\}$, and $T = 10^8$ runs.

Obviously, increasing the noise levels of all experts ($v_a = v_b = v_c$) decreases the cardinality of the set where the always starting C defeats AB. If $v_a = v_b = v_c = \frac{1}{8}$, AB loses for $k \in \{2, 4, \ldots, 24, 48\}$. This is a little more than half of the considered selection numbers. In contrast, C defeats AB for $v_a = v_b = v_c = 1$ only for $k = 2$.

Now expert A and expert B are still equally good and expert C is better than both A and B (case (2)). We consider $v_a = v_b = \frac{1}{4}$ and $v_c = \frac{1}{8}$ and the results are figured out in Figure 6.3.

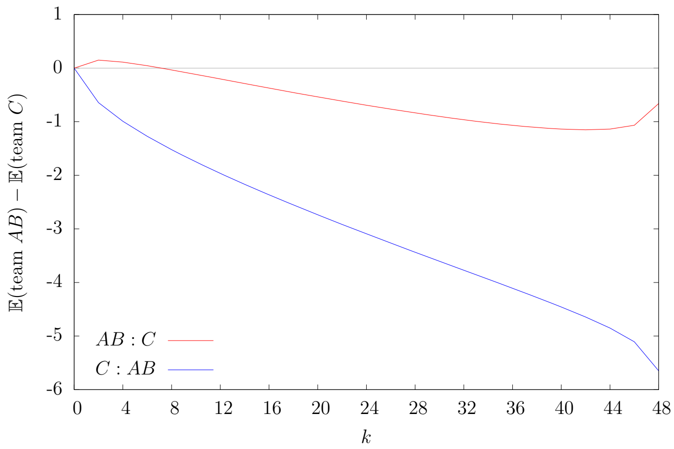


Figure 6.3 Scenarios $AB : C$ and $C : AB$ for $n = 96$, $v_a = v_b = \frac{1}{4}$, $v_c = \frac{1}{8}$, and $T = 10^8$ runs.

For these noise levels we now get reverse results as for $v_a = v_b = v_c = \frac{1}{4}$. For scenario $AB : C$ the double-expert-team AB only wins for $k \in \{2, 4, 6\}$. In all other cases the single-expert-team C wins.

If now $v_a < v_b$ (cases (3) and (4)), in addition to $AB : C$ and $C : AB$ we also consider scenarios $BA : C$ and $C : BA$. Firstly, we analyse case (3), where experts A and C are equally good and B is worth than both A and B. Therefore Figure 6.4 illustrates the results of the scenarios mentioned above for $n = 96$, $v_a = v_c = \frac{1}{4}$ and $v_b = \frac{1}{2}$.

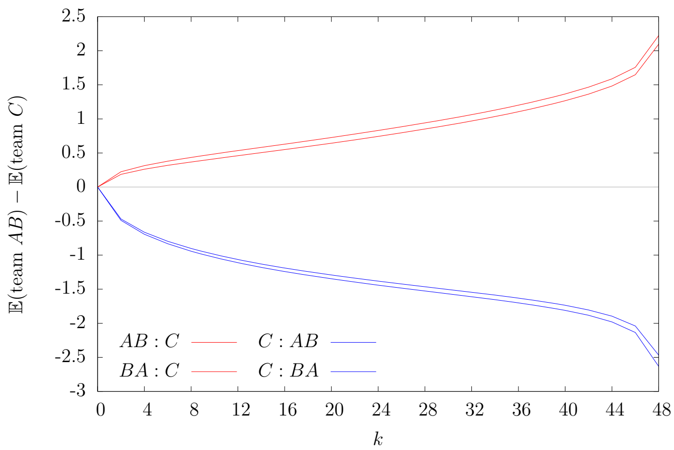


Figure 6.4 Scenarios $AB : C$, $BA : C$, $C : AB$, and $C : BA$ for $n = 96$, $v_a = v_c = \frac{1}{4}$, $v_b = \frac{1}{2}$, and $T = 10^8$ runs.

Considering both C-starting scenarios $C : AB$ and $C : BA$, team C wins for all considered k independently of the selection orders of A and B. On the other hand, team AB always wins for both AB-starting scenarios $AB : C$ and $BA : C$. Analysing the selection orders of team AB, we observe that order AB performs better against C than order BA does. This observation comes true for AB- as well as C-starting scenarios. Further, by increasing $v_b - v_a$ the difference between the orders AB and BA increases. We executed an experiment for $v_a = \frac{1}{8}$, $v_b = 1$, and $v_c = \frac{1}{4}$ (case (4)) in Figure 6.5.

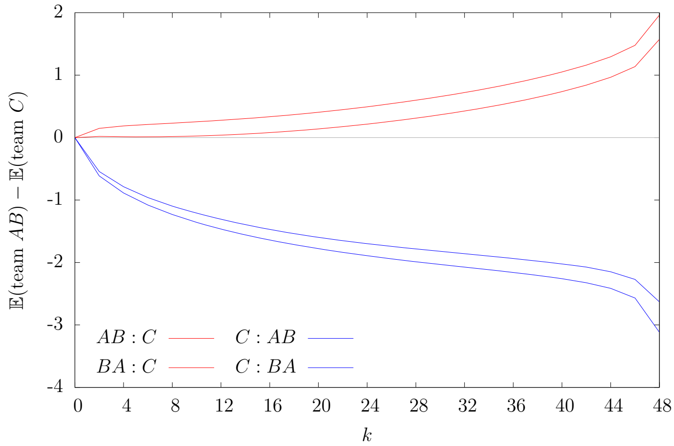


Figure 6.5 Scenarios $AB : C$, $BA : C$, $C : AB$, and $C : BA$ for $n = 96$, $v_a = \frac{1}{8}$, $v_b = 1$, $v_c = \frac{1}{4}$, and $T = 10^8$ runs.

As mentioned above, the difference between the results of team AB with order AB and team C ($\mathbb{E}$("team AB with order AB")$-\mathbb{E}$("team C")) is obviously larger than $\mathbb{E}$("team AB with order BA")$-\mathbb{E}$("team C"). So, order AB performs better against team C. Regarding the noise levels, orders AB and BA win for the AB-starting scenarios and both loose for the C-starting scenarios.

To comprehend the results of TeSeP with equally good experts A, B, and C, team AB is better than team C with scenario $AB : C$ for all considered noise levels $v_a = v_b = v_c \in \{\frac{1}{8}, \frac{1}{4}, \frac{1}{2}, 1\}$. Considering a fair competition of this scenario ($\frac{T}{2}$ times order $ACBC$ and $\frac{T}{2}$ times order $CACB$), team AB is also better than team C for these noise levels. Both observations are figured out in Appendix C (see Figures C.1 and C.2) on page 110.

If experts A, B, or C have different noise levels (cases (2), (3), and (4)), within the double-expert-team the better expert A should always start selecting. So, in all of our experiment $AB : C$ is better than $BA : C$ and $C : AB$ is better than $C : BA$.

7 Results of the Group Selection Problem

Explaining both experimental and theoretical results of the Group Selection Problem with two, three, and four experts is the main focus of this chapter. The chapter is divided into three sections each including the results of one model of GSeP. There are additional results presented in Appendix D starting on page 111. All applied algorithms are implemented using programming language Java™. Each experiment is run with $T = 10^p$ ($p \in \{8, 9\}$) simulations and the results have p decimals.

In contrast to 2-SeP (see Chapter 4 starting on page 35) and TeSeP (see Chapter 6 starting on page 51) we do not consider a uniform set of the selection numbers. Ensuring each expert selects the same number of items (as his team member), we suppose

- $k_1, k_2 \in \{0, 1, 2, \ldots, n\}$ for 2-GSeP,
- $k \in \{0, 2, 4, \ldots, n\}$ for 3-GSeP,
- $k \in \{0, 2, 4, \ldots, n\}$ for 4-GSeP with scenarios of $(2;2)$-type, and
- $k \in \{0, 3, 6, \ldots, n\}$ for 4-GSeP with scenarios of $(1;3)$-type.

7.1 The Group Selection Problem with Two Experts

The first section of this chapter contains experimental results of 2-GSeP. As defined in Model 2.4.1 on page 23, the noise level of expert A is smaller than the noise level of expert B always. Formally, $v_a < v_b$. Considering n items and k_1 and k_2 ($k_1 < k_2 \leq n$) items to select we now ask: Which expert should select which number of items? Or: Should expert A or expert B select the smaller (larger) number of items? Answering these two questions, Figure 7.1.1 illustrates the absolute performance differences between both scenarios $A_{k_1}; B_{k_2}$ and $A_{k_2}; B_{k_1}$ for $n = 96$, $v_a = \frac{1}{4}$, and $v_b = \frac{1}{2}$.

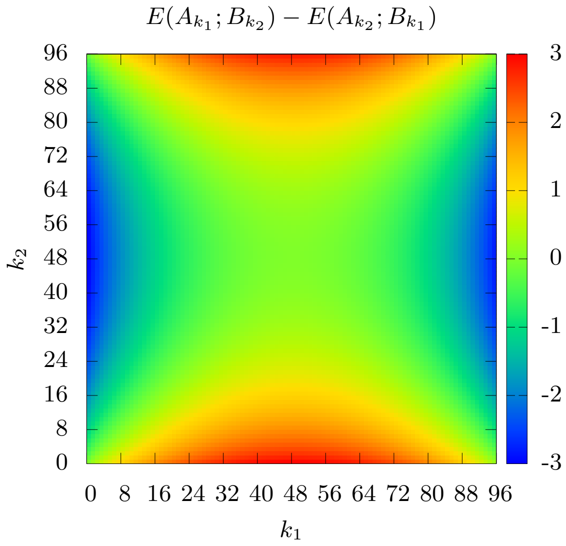


Figure 7.1.1 Absolute performance differences between scenarios $A_{k_1}; B_{k_2}$ and $A_{k_2}; B_{k_1}$ for $n = 96$, $v_a = \frac{1}{4}$, $v_b = \frac{1}{2}$, and $T = 10^8$ runs.

Obviously, there are three different cases:

(1) $A_{k_1}; B_{k_2}$ is better than $A_{k_2}; B_{k_1}$ for
$\{(k_1, k_2) : (k_1 + k_2 < n \wedge k_1 > k_2) \vee (k_1 + k_2 > n \wedge k_1 < k_2)\}$
(indicated by red to yellow area)

(2) $A_{k_1}; B_{k_2}$ is worse than $A_{k_2}; B_{k_1}$ for
$\{(k_1, k_2) : (k_1 + k_2 < n \wedge k_1 < k_2) \vee (k_1 + k_2 > n \wedge k_1 > k_2)\}$
(indicated by blue to turquoise area)

(3) $A_{k_1}; B_{k_2}$ and $A_{k_2}; B_{k_1}$ are equally good for
$\{(k_1, k_2) : k_1 + k_2 = n \vee k_1 = k_2\}$

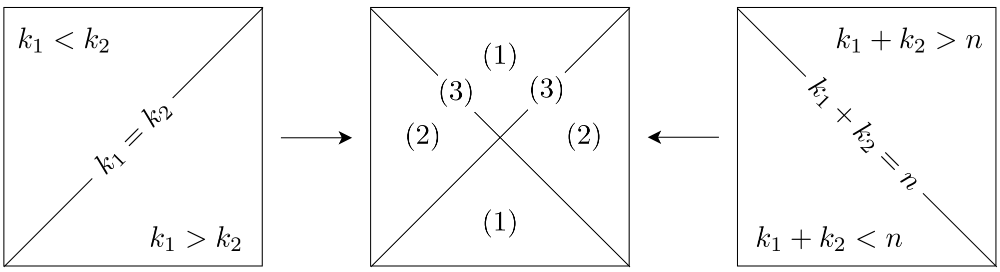


Figure 7.1.2 Typical segments in the resulted plots of 2-GSeP.

Whereas case (3) represents both diagonals from the lower left to the upper right corner and from the upper left to the lower right corner, cases (1) and (2) both represent an intersection of each two large triangles. Both intersections of the upper or lower large triangles describe case (1). And case (2) is expressed by the intersection of the upper and the lower large triangles on the left side as well as the intersection of the lower and upper large triangles on the right side.

7.2 The Group Selection Problem with Three Experts

In the first part of this section we present some experimental results of 3-GSeP. After these insights we give an overview of how all considered scenarios ranked among each other (see Figure 7.2.8 on page 61). Following this, the second part of this section contains theoretical results of 3-GSeP for $n = 3$ and $k = 2$.

As introduced in Section 2.4 starting on page 22, there are six different scenarios: $A;BC$ and $A;CB$, $B;AC$ and $B;CA$, and also $C;AB$ and $C;BA$. Altogether, there are $\frac{6 \cdot 5}{2} = 15$ relations. In the first step we show the absolute performance differences between each two scenarios involving the single-expert-group A (red), B (green), or C (blue).

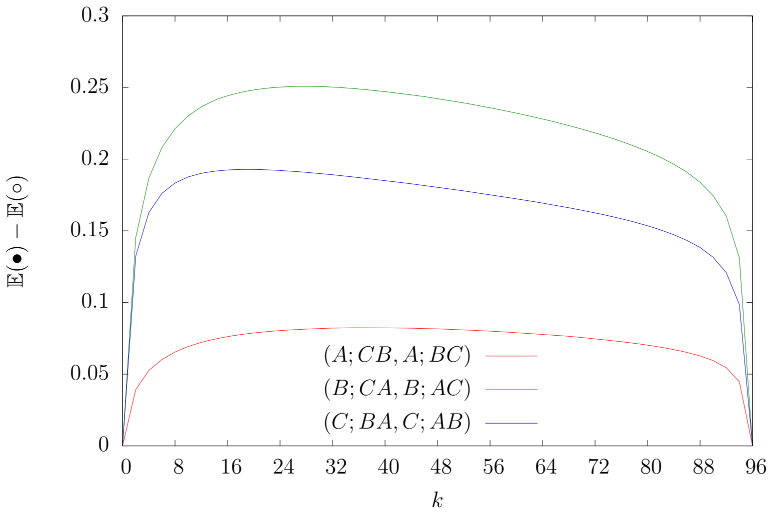


Figure 7.2.1 Absolute performance differences between scenarios $A;CB$ and $A;BC$, $B;CA$ and $B;AC$, and also $C;BA$ and $C;AB$ for $n = 96$, $v_a = \frac{1}{16}$, $v_b = \frac{1}{2}$, $v_c = 1$, and $T = 10^8$ runs.

Within the red scenarios $A;CB$ is better than $A;BC$. So, in the BC-group the worse expert C should start selecting. Among $B;CA$ and $B;AC$ or $C;BA$ and $C;AB$ there is the same effect. Scenarios starting with a selection action of the worse expert are better than those starting with a selection action of the better expert. These resulted relations are marked in Figure 7.2.8 on page 61 using "(1)".

Whereas the absolute performance differences between scenarios $B;CA$ and $B;AC$

stay constant for different noise levels $\frac{1}{16} < v_b < 1$, both $\mathbb{E}(A;CB) - \mathbb{E}(A;BC)$ and $\mathbb{E}(C;BA) - \mathbb{E}(C;AB)$ differ dependent on v_b. Increasing v_b decreases $\mathbb{E}(A;CB) - \mathbb{E}(A;BC)$ and increases $\mathbb{E}(C;BA) - \mathbb{E}(C;AB)$. Figure 7.2.2 and Figure 7.2.3 show these relations.

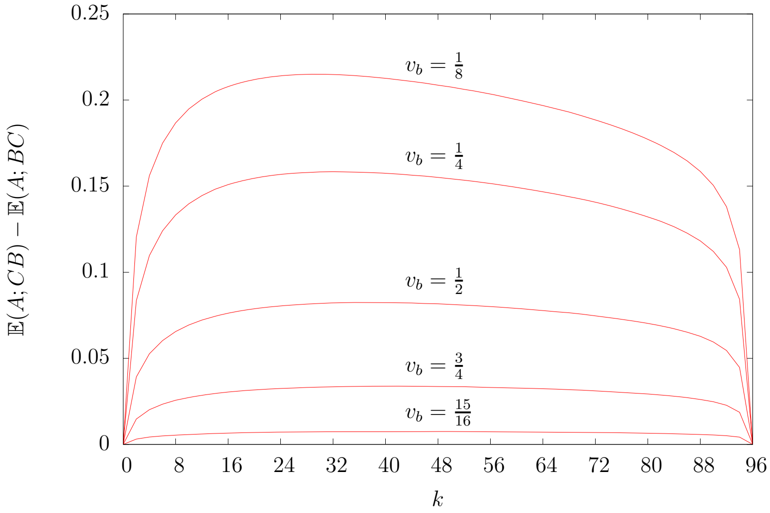


Figure 7.2.2 Absolute performance differences between scenarios $A;CB$ and $A;BC$ for $n = 96$, $v_a = \frac{1}{16}$, $v_b \in \{\frac{1}{8}, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, \frac{15}{16}\}$, $v_c = 1$, and $T = 10^8$ runs.

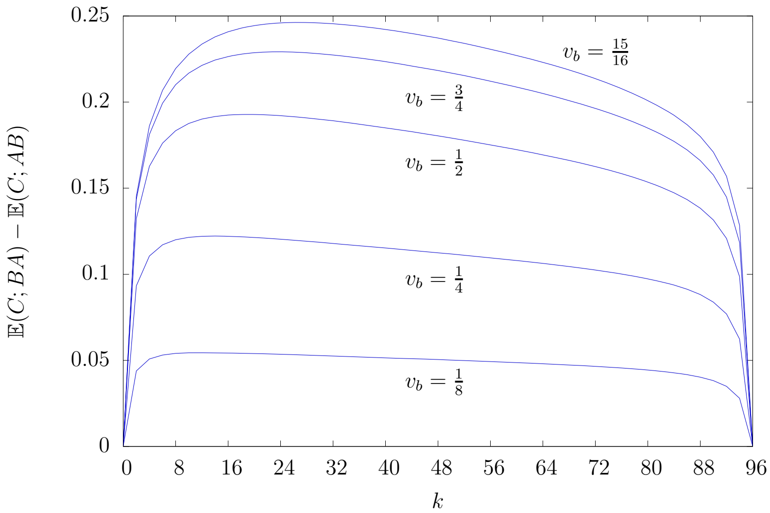


Figure 7.2.3 Absolute performance differences between scenarios $C;BA$ and $C;AB$ for $n = 96$, $v_a = \frac{1}{16}$, $v_b \in \{\frac{1}{8}, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, \frac{15}{16}\}$, $v_c = 1$, and $T = 10^8$ runs.

After we have seen the absolute performance differences between each two scenarios involving the single-expert-group A, B, or C we are now interested in the ranking among the winners ($A;CB$, $B;CA$, $C;BA$) and the losers ($A;BC$, $B;AB$, $C;AB$). The results are presented below in Figure 7.2.4.

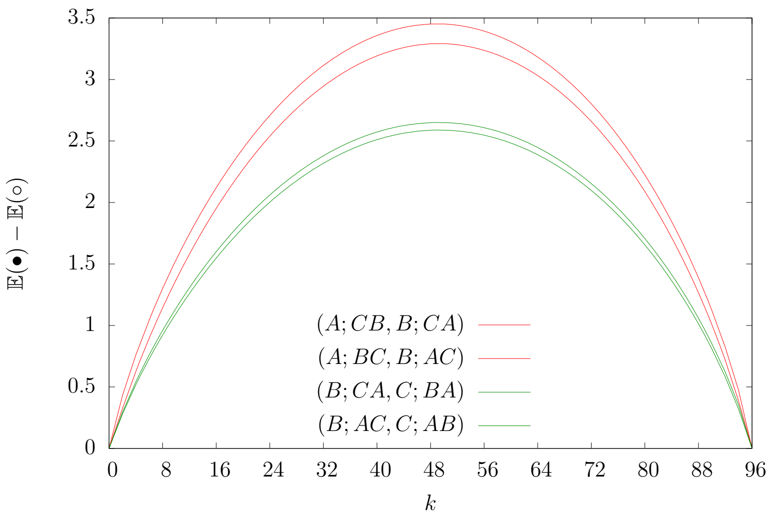


Figure 7.2.4 Absolute performance differences between the winners and losers as a result of Figure 7.2.1 for $n = 96$, $v_a = \frac{1}{16}$, $v_b = \frac{1}{2}$, $v_c = 1$, and $T = 10^8$ runs.

Obviously, among the winners (solid lines) $A;CB$ is better than $B;CA$ and $B;CA$ is better than $C;BA$. As a consequence, $A;CB$ is better than $C;BA$, too. The losers (dashed lines) are related analogously. So, $A;BC$ is better than $B;AC$, $B;AC$ is better than $C;AB$, and $A;BC$ is better than $C;AB$ consequently. These six relations are marked in Figure 7.2.8 on page 61 using "(2)" and "(2′)". With (1) and (2) three additional relations can be defined. They are marked with "(3)". So, there are three relations left. Firstly, in Figure 7.2.5 we compare $A;BC$ with $B;CA$ (red) and $B;AC$ with $C;BA$ (green) for $v_b = \frac{1}{8}$ (solid lines) and $v_b = \frac{15}{16}$ (dashed lines).

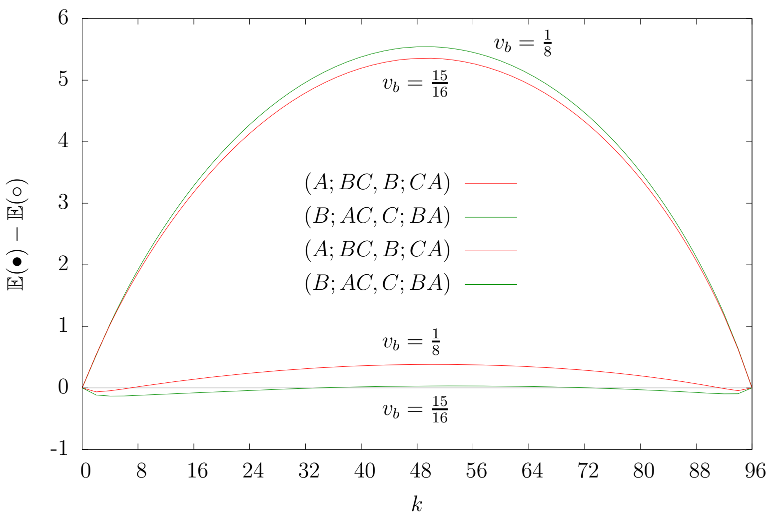


Figure 7.2.5 Absolute performance differences between scenarios $A;BC$ and $B;CA$ and also $B;AC$ and $C;BA$ for $n = 96$, $v_a = \frac{1}{16}$, $v_b \in \{\frac{1}{8}, \frac{15}{16}\}$, $v_c = 1$, and $T = 10^8$ runs.

Figure 7.2.5 shows that we can not rank $A;BC$ and $B;CA$ or $B;AC$ and $C;BA$. For $v_b = \frac{1}{8}$ the absolute performance differences between $A;BC$ and $B;CA$ are not uniformly larger than zero. This effect also happens to the relation between scenarios $B;AC$ and $C;BA$ for $v_b = \frac{15}{16}$. For that reason, Figure 7.2.6 (enlarged Figure 7.2.5) gives closer insights.

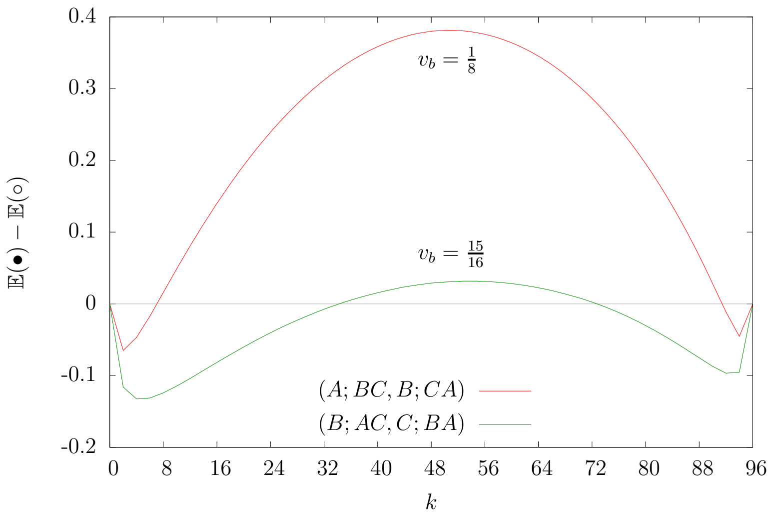


Figure 7.2.6 Absolute performance differences between scenarios $A;BC$ and $B;CA$ and also $B;AC$ and $C;BA$ for $n = 96$, $v_a = \frac{1}{16}$, $v_b \in \{\frac{1}{8}, \frac{15}{16}\}$, $v_c = 1$, and $T = 10^8$ runs.

These two relations are marked in Figure 7.2.8 on page 61 using "(4)". So, there is only one relation, $A;BC$ versus $C;BA$, left. Figure 7.2.7 presents the results for $v_b \in \{\frac{1}{8}, \frac{1}{2}, \frac{15}{16}\}$.

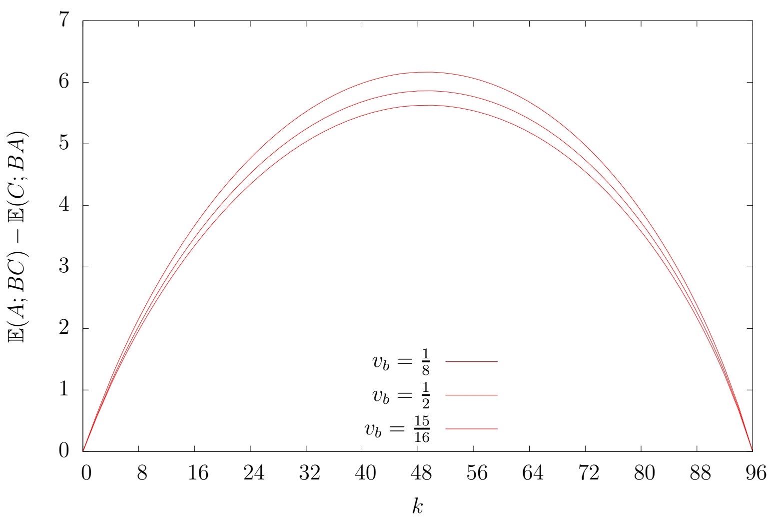


Figure 7.2.7 Absolute performance differences between scenarios $A;BC$ and $C;BA$ for $n = 96$, $v_a = \frac{1}{16}$, $v_b \in \{\frac{1}{8}, \frac{1}{2}, \frac{15}{16}\}$, $v_c = 1$, and $T = 10^8$ runs.

Obviously, $A;BC$ is better than $C;BA$ for all considered v_b. Finally, this relation is also marked in Figure 7.2.8 below. In this case we use "(5)" for the notation.

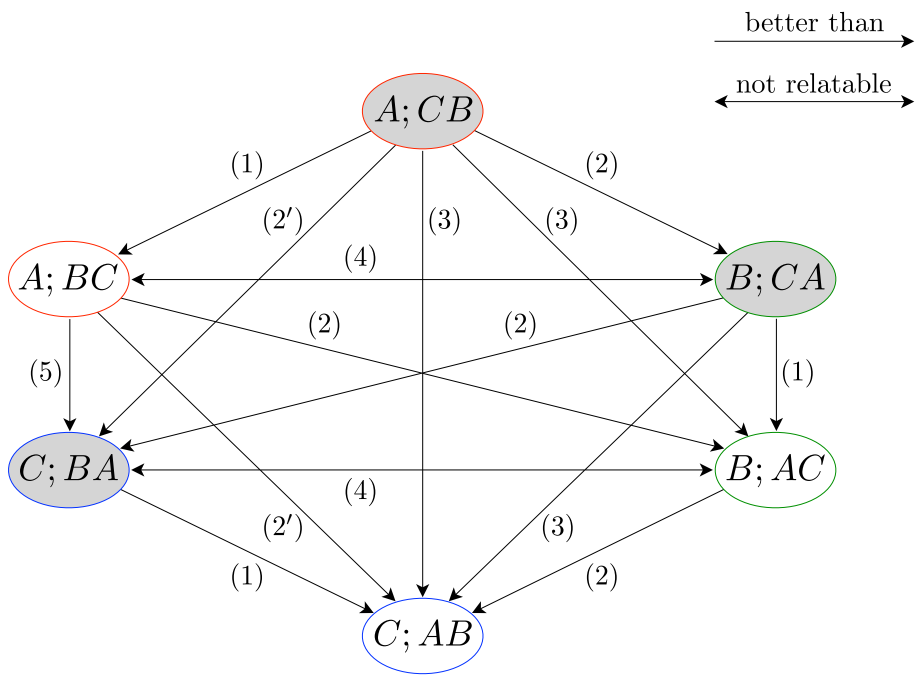


Figure 7.2.8 Ranking of all 3-GSeP-scenarios for $n = 96$, $v_a = \frac{1}{16}$, $v_b \in \{\frac{1}{8}, \frac{1}{2}, \frac{15}{16}\}$, $v_c = 1$, and $T = 10^8$ runs. Figure 7.2.1 ⇒ (1), Figure 7.2.4 ⇒ (2) ⇒ (2′), (1) and (2) ⇒ (3), Figure 7.2.5 ⇒ (4), and Figure 7.2.7 ⇒ (5).

After we have drawn all 15 relations there are two main observations. Firstly, scenario $A;CB$ is better than any other scenario and secondly, scenario $C;AB$ is worse than any other scenario. Beyond that, rank two and three are shared by $A;BC$ and $B;CA$. And rank four and five are shared by $C;BA$ and $B;AC$.

As already announced, the second part of this section contains theoretical results for $n = 3$ and $k = 2$ based on these experimental results. For this purpose, we use the model of permutations introduced by Bärthel ([Bae2011]) in her report (see Section 3.4 starting on page 31). So, we need two different values to proof each of the relations presented above: $P_A(3, 2, \{1, 2\})$ and $P_{AB}(3, 2, \{1, 2\})$.

Remark Bärthel defined the Assortment Problem as a **minimisation problem**. In detail, $x_1 < x_2 < \ldots < x_n$ and $1 := x_1,\ 2 := x_2,\ \ldots,\ n := x_n$. In 3-GSeP (as one kind of the Selection Problem) we define a **maximisation problem**. Here, $x_1 > x_2 > \ldots > x_n$ and $1 := x_1,\ 2 := x_2,\ \ldots,\ n := x_n$.

$P_A(3, 2, \{1, 2\})$ is the probability of expert A selecting $k = 2$ out of $n = 3$ items including the best (= largest) item 1 and the second best item 2. There are only two possible permutations to select these items: $\pi = (1,2,3)$ and $\pi = (2,1,3)$. For $\pi = (1,2,3)$ expert A observes the items in the right order. So, the number of inversions $l(\pi) = l(1,2,3) = \#\{(i,j) \in \{1,2,3\}^2 : i < j, \pi(i) > \pi(j)\} = 0$ and expert A selects item 1 and item 2. For $\pi = (2,1,3)$ expert A observes $x_2 + a_2 > x_1 + a_1 > x_3 + a_3$ and so $l(\pi) = l(2,1,3) = 1$. In this case expert A firstly selects item 2 and secondly selects item 1. $P_A(3,2,\{1,2\})$ is composed as follows:

$$P_A(3,2,\{1,2\}) = \frac{\alpha^{l(1,2,3)}}{p_3(\alpha)} + \frac{\alpha^{l(2,1,3)}}{p_3(\alpha)} = \frac{\alpha^0}{p_3(\alpha)} + \frac{\alpha^1}{p_3(\alpha)} = \frac{p_2(\alpha)}{p_3(\alpha)} > 0$$

$P_{AB}(3,2,\{1,2\})$ is the probability of experts A and B selecting $k = 2$ out of $n = 3$ items including item 1 and item 2. In contrast to $P_A(3,2,\{1,2\})$, we need two double steps now. The first double step means, expert A selects item 1 and expert B selects item 2. And the second double step means, expert A selects item 2 and expert B selects item 1.

First double step:

A observes $x_1 + a_1 > x_2 + a_2 > x_3 + a_3$
or $x_1 + a_1 > x_3 + a_3 > x_2 + a_2$ A selects item 1

and B observes $x_1 + b_1 > x_2 + b_2 > x_3 + b_3$
or $x_2 + b_2 > x_1 + b_1 > x_3 + b_3$
or $x_2 + b_2 > x_3 + b_3 > x_1 + b_1$ B selects item 2

For both permutations $\pi \in \{(1,2,3),(1,3,2)\}$ expert A selects item 1 because he believes in item 1 as the largest item. In this way expert B selects item 2 for $\pi = \{(2,1,3),(2,3,1)\}$, too. For $\pi = (1,2,3)$ expert B believes in item 1 as the largest item. But he selects his second best item (= 2) because of expert A has already selected item 1. So, items 1 and 2 are selected.

Second double step:

A observes $x_2 + a_2 > x_1 + a_1 > x_3 + a_3$
or $x_2 + a_2 > x_3 + a_3 > x_1 + a_1$ A selects item 2

and B observes $x_1 + b_1 > x_2 + b_2 > x_3 + b_3$
or $x_1 + b_1 > x_3 + b_3 > x_2 + b_2$
or $x_2 + b_2 > x_1 + b_1 > x_3 + b_3$ B selects item 1

Here, expert A selects item 2 for $\pi \in \{(2,1,3),(2,3,1)\}$ and expert B selects item 1 for $\pi \in \{(1,2,3),(1,3,2)\}$. As explained above, expert B believes in item 2 as the largest item for $\pi = (2,1,3)$. But he selects item 1 because item 2 has already been selected by expert A. After describing these two double steps $P_{AB}(3,2,\{1,2\})$ is composed as follows:

$$
\begin{aligned}
P_{AB}(3,2,\{1,2\}) &= \underbrace{\left(\frac{\alpha^{l(1,2,3)}}{p_3(\alpha)} + \frac{\alpha^{l(1,3,2)}}{p_3(\alpha)}\right) \cdot \left(\frac{\beta^{l(1,2,3)}}{p_3(\beta)} + \frac{\beta^{l(2,1,3)}}{p_3(\beta)} + \frac{\beta^{l(2,3,1)}}{p_3(\beta)}\right)}_{\text{first double step}} + \\
&\underbrace{\left(\frac{\alpha^{l(2,1,3)}}{p_3(\alpha)} + \frac{\alpha^{l(2,3,1)}}{p_3(\alpha)}\right) \cdot \left(\frac{\beta^{l(1,2,3)}}{p_3(\beta)} + \frac{\beta^{l(1,3,2)}}{p_3(\beta)} + \frac{\beta^{l(2,1,3)}}{p_3(\beta)}\right)}_{\text{second double step}} \\
&= \left(\frac{\alpha^0}{p_3(\alpha)} + \frac{\alpha^1}{p_3(\alpha)}\right) \cdot \left(\frac{\beta^0}{p_3(\beta)} + \frac{\beta^1}{p_3(\beta)} + \frac{\beta^2}{p_3(\beta)}\right) + \\
&\left(\frac{\alpha^1}{p_3(\alpha)} + \frac{\alpha^2}{p_3(\alpha)}\right) \cdot \left(\frac{\beta^0}{p_3(\beta)} + \frac{\beta^1}{p_3(\beta)} + \frac{\beta^1}{p_3(\beta)}\right) \\
&= \frac{p_2(\alpha)}{p_3(\alpha)} \cdot \frac{p_1(\beta) + \beta p_2(\beta)}{p_3(\beta)} + \frac{\alpha p_2(\alpha)}{p_3(\alpha)} \cdot \frac{\beta p_1(\beta) + p_2(\beta)}{p_3(\beta)} \\
&= \frac{p_2(\alpha)}{p_3(\alpha)} \cdot \frac{p_1(\beta) + \beta\, p_2(\beta) + \alpha \cdot [\beta\, p_1(\beta) + p_2(\beta)]}{p_3(\beta)}
\end{aligned}
$$

With $P_A(3,2,\{1,2\})$ and $P_{AB}(3,2,\{1,2\})$ we can establish these values for the groups B, C, BA, AC, CA, BC, and CB analogously. To present theoretical results for the relations between all six scenarios of 3-GSeP, we define the probability of group A and group BC (with selection order BC) each selecting the two largest out of three items as

$$
\begin{aligned}
P_{A;BC}(3,2,\{1,2\}) &:= P_{A;BC}(3,2,\{1,2\};3,2,\{1,2\}) \\
&= P_A(3,2,\{1,2\}) \cdot P_{BC}(3,2,\{1,2\}).
\end{aligned}
$$

This equation is true because the selecting process of group A is independent of the selecting process of group BC. After presenting experimental results for the relations between each two scenarios involving the single-expert-group A, B, and C for $n = 96$ in Figure 7.2.1 on page 57, we show accordingly theoretical results for $n = 3$ and $k = 2$ now. They are marked with “(1)” in Figure 7.2.8 on page 61.

Theorem 7.2.1 *For all $\alpha, \beta, \gamma \in (0,1)$ with $\alpha < \beta < \gamma$ are*

(i) $P_{A;CB}(3,2,\{1,2\}) > P_{A;BC}(3,2,\{1,2\})$,

(ii) $P_{B;CA}(3,2,\{1,2\}) > P_{B;AC}(3,2,\{1,2\})$, *and*

(iii) $P_{C;BA}(3,2,\{1,2\}) > P_{C;AB}(3,2,\{1,2\})$.

Proof. (*i*) $P_{A;CB}(3,2,\{1,2\}) > P_{A;BC}(3,2,\{1,2\})$

$$\iff \quad P_A(3,2,\{1,2\}) \cdot P_{CB}(3,2,\{1,2\}) > P_A(3,2,\{1,2\}) \cdot P_{BC}(3,2,\{1,2\})$$
$$\iff \quad P_{CB}(3,2,\{1,2\}) > P_{BC}(3,2,\{1,2\})$$

(*ii*) $P_{B;AC}(3,2,\{1,2\}) > P_{B;CA}(3,2,\{1,2\})$

$$\iff \quad P_B(3,2,\{1,2\}) \cdot P_{CA}(3,2,\{1,2\}) > P_B(3,2,\{1,2\}) \cdot P_{AC}(3,2,\{1,2\})$$
$$\iff \quad P_{CA}(3,2,\{1,2\}) > P_{AC}(3,2,\{1,2\})$$

(*iii*) $P_{C;BA}(3,2,\{1,2\}) > P_{C;AB}(3,2,\{1,2\})$

$$\iff \quad P_C(3,2,\{1,2\}) \cdot P_{BA}(3,2,\{1,2\}) > P_C(3,2,\{1,2\}) \cdot P_{AB}(3,2,\{1,2\})$$
$$\iff \quad P_{BA}(3,2,\{1,2\}) > P_{AB}(3,2,\{1,2\})$$

With $\alpha, \beta, \gamma \in (0,1)$, $\alpha < \beta < \gamma$ and Theorem 3.4.2 on page 33 each of these inequalities is true.

□

The next theorem shows how the winners ($A;CB$, $B;CA$, $C;BA$) and the losers ($A;BC$, $B;AB$, $C;AB$) of Theorem 7.2.1 on page 63 are ranked among each other. In Figure 7.2.8 on page 61 they are marked with "(2)" and "(2′)".

Theorem 7.2.2 *For all $\alpha, \beta, \gamma \in (0,1)$ with $\alpha < \beta < \gamma$ are*

(*i*) $P_{A;CB}(3,2,\{1,2\}) > P_{B;CA}(3,2,\{1,2\}) > P_{C;BA}(3,2,\{1,2\})$ *and*

(*ii*) $P_{A;BC}(3,2,\{1,2\}) > P_{B;AC}(3,2,\{1,2\}) > P_{C;AB}(3,2,\{1,2\})$.

Proof. (*i*) $P_{A;CB}(3,2,\{1,2\}) > P_{B;CA}(3,2,\{1,2\}) > P_{C;BA}(3,2,\{1,2\})$

$$P_{A;CB}(3,2,\{1,2\}) > P_{B;CA}(3,2,\{1,2\})$$

$$\iff \quad P_A(3,2,\{1,2\}) \cdot P_{CB}(3,2,\{1,2\}) > P_B(3,2,\{1,2\}) \cdot P_{CA}(3,2,\{1,2\})$$

$$\iff \quad \frac{p_2(\alpha)}{p_3(\alpha)} \cdot \frac{p_2(\gamma)}{p_3(\gamma)} \cdot \frac{p_1(\beta) + \beta\, p_2(\beta) + \gamma \cdot [\beta\, p_1(\beta) + p_2(\beta)]}{p_3(\beta)} >$$
$$\frac{p_2(\beta)}{p_3(\beta)} \cdot \frac{p_2(\gamma)}{p_3(\gamma)} \cdot \frac{p_1(\alpha) + \alpha\, p_2(\alpha) + \gamma \cdot [\alpha\, p_1(\alpha) + p_2(\alpha)]}{p_3(\alpha)}$$

$$\iff \quad p_2(\alpha) \cdot (p_1(\beta) + \beta\, p_2(\beta) + \gamma \cdot [\beta\, p_1(\beta) + p_2(\beta)]) >$$
$$p_2(\beta) \cdot (p_1(\alpha) + \alpha\, p_2(\alpha) + \gamma \cdot [\alpha\, p_1(\alpha) + p_2(\alpha)])$$

$$\iff \quad (1+\alpha) \cdot (1 + \beta\,(1+\beta) + \gamma \cdot [\beta + (1+\beta)]) >$$
$$(1+\beta) \cdot (1 + \alpha\,(1+\alpha) + \gamma \cdot [\alpha + (1+\alpha)])$$

$$\iff \quad (\alpha + \alpha\beta + \beta + \gamma) \cdot (\beta - \alpha) > 0$$

With $\alpha < \beta$ this inequality is true.

$$
\begin{aligned}
& P_{B;CA}(3,2,\{1,2\}) > P_{C;BA}(3,2,\{1,2\}) \\
\iff\ & P_B(3,2,\{1,2\}) \cdot P_{CA}(3,2,\{1,2\}) > P_C(3,2,\{1,2\}) \cdot P_{BA}(3,2,\{1,2\}) \\
\iff\ & \frac{p_2(\beta)}{p_3(\beta)} \cdot \frac{p_2(\gamma)}{p_3(\gamma)} \cdot \frac{p_1(\alpha) + \alpha\, p_2(\alpha) + \gamma \cdot [\alpha\, p_1(\alpha) + p_2(\alpha)]}{p_3(\alpha)} > \\
& \frac{p_2(\gamma)}{p_3(\gamma)} \cdot \frac{p_2(\beta)}{p_3(\beta)} \cdot \frac{p_1(\alpha) + \alpha\, p_2(\alpha) + \beta \cdot [\alpha\, p_1(\alpha) + p_2(\alpha)]}{p_3(\alpha)} \\
\iff\ & \gamma > \beta
\end{aligned}
$$

(ii) $P_{A;BC}(3,2,\{1,2\}) > P_{B;AC}(3,2,\{1,2\}) > P_{C;AB}(3,2,\{1,2\})$

$$
\begin{aligned}
& P_{A;BC}(3,2,\{1,2\}) > P_{B;AC}(3,2,\{1,2\}) \\
\iff\ & P_A(3,2,\{1,2\}) \cdot P_{BC}(3,2,\{1,2\}) > P_B(3,2,\{1,2\}) \cdot P_{AC}(3,2,\{1,2\}) \\
\iff\ & \frac{p_2(\alpha)}{p_3(\alpha)} \cdot \frac{p_2(\beta)}{p_3(\beta)} \cdot \frac{p_1(\gamma) + \gamma\, p_2(\gamma) + \beta \cdot [\gamma\, p_1(\gamma) + p_2(\gamma)]}{p_3(\gamma)} > \\
& \frac{p_2(\beta)}{p_3(\beta)} \cdot \frac{p_2(\alpha)}{p_3(\alpha)} \cdot \frac{p_1(\gamma) + \gamma\, p_2(\gamma) + \alpha \cdot [\gamma\, p_1(\gamma) + p_2(\gamma)]}{p_3(\gamma)} \\
\iff\ & \beta > \alpha
\end{aligned}
$$

$$
\begin{aligned}
& P_{B;AC}(3,2,\{1,2\}) > P_{C;AB}(3,2,\{1,2\}) \\
\iff\ & P_B(3,2,\{1,2\}) \cdot P_{AC}(3,2,\{1,2\}) > P_C(3,2,\{1,2\}) \cdot P_{AB}(3,2,\{1,2\}) \\
\iff\ & \frac{p_2(\beta)}{p_3(\beta)} \cdot \frac{p_2(\alpha)}{p_3(\alpha)} \cdot \frac{p_1(\gamma) + \gamma\, p_2(\gamma) + \alpha \cdot [\gamma\, p_1(\gamma) + p_2(\gamma)]}{p_3(\gamma)} > \\
& \frac{p_2(\gamma)}{p_3(\gamma)} \cdot \frac{p_2(\alpha)}{p_3(\alpha)} \cdot \frac{p_1(\beta) + \beta\, p_2(\beta) + \alpha \cdot [\beta\, p_1(\beta) + p_2(\beta)]}{p_3(\beta)} \\
\iff\ & p_2(\beta) \cdot (p_1(\gamma) + \gamma\, p_2(\gamma) + \alpha \cdot [\gamma\, p_1(\gamma) + p_2(\gamma)]) > \\
& p_2(\gamma) \cdot (p_1(\beta) + \beta\, p_2(\beta) + \alpha \cdot [\beta\, p_1(\beta) + p_2(\beta)]) \\
\iff\ & (1+\beta) \cdot (1 + \gamma \cdot (1+\gamma) + \alpha \cdot [\gamma + (1+\gamma)]) > \\
& (1+\gamma) \cdot (1 + \beta \cdot (1+\beta) + \alpha \cdot [\beta + (1+\beta)]) \\
\iff\ & (\alpha + \beta + \beta\gamma + \gamma) \cdot (\gamma - \beta) > 0
\end{aligned}
$$

□

The next theorem presents theoretical results for all leftover ”better than”-relations. They are marked with “(3)” and “(5)” in Figure 7.2.8 on page 61.

Theorem 7.2.3 *For all* $\alpha, \beta, \gamma \in (0,1)$ *with* $\alpha < \beta < \gamma$ *are*

(i) $P_{A;BC}(3,2,\{1,2\}) > P_{C;BA}(3,2,\{1,2\})$,

(ii) $P_{A;CB}(3,2,\{1,2\}) > P_{C;AB}(3,2,\{1,2\})$,

(iii) $P_{A;CB}(3,2,\{1,2\}) > P_{B;AC}(3,2,\{1,2\})$, *and*

(iv) $P_{B;CA}(3,2,\{1,2\}) > P_{C;AB}(3,2,\{1,2\})$.

Proof. (i) $P_{A;BC}(3,2,\{1,2\}) > P_{C;BA}(3,2,\{1,2\})$

$$\iff \quad P_A(3,2,\{1,2\}) \cdot P_{BC}(3,2,\{1,2\}) > P_C(3,2,\{1,2\}) \cdot P_{BA}(3,2,\{1,2\})$$

$$\iff \quad \frac{p_2(\alpha)}{p_3(\alpha)} \cdot \frac{p_2(\beta)}{p_3(\beta)} \cdot \frac{p_1(\gamma) + \gamma p_2(\gamma) + \beta \cdot [\gamma p_1(\gamma) + p_2(\gamma)]}{p_3(\gamma)} > \frac{p_2(\gamma)}{p_3(\gamma)} \cdot \frac{p_2(\beta)}{p_3(\beta)} \cdot \frac{p_1(\alpha) + \alpha p_2(\alpha) + \beta \cdot [\alpha p_1(\alpha) + p_2(\alpha)]}{p_3(\alpha)}$$

$$\iff \quad p_2(\alpha) \cdot (p_1(\gamma) + \gamma p_2(\gamma) + \beta \cdot [\gamma p_1(\gamma) + p_2(\gamma)]) > p_2(\gamma) \cdot (p_1(\alpha) + \alpha p_2(\alpha) + \beta \cdot [\alpha p_1(\alpha) + p_2(\alpha)])$$

$$\iff \quad (1+\alpha) \cdot (1 + \gamma(1+\gamma) + \beta \cdot [\gamma + (1+\gamma)]) > (1+\gamma) \cdot (1 + \alpha(1+\alpha) + \beta \cdot [\alpha + (1+\alpha)])$$

$$\iff \quad (\alpha \cdot (1+\gamma) + \beta + \gamma) \cdot (\gamma - \alpha) > 0$$

(ii) With Theorem 7.2.1 (ii) and Theorem 7.2.2 this inequality is true.

(iii) With Theorem 7.2.1 (ii) and Theorem 7.2.2 (i) this inequality is true.

(iv) With Theorem 7.2.1 (ii) and Theorem 7.2.2 (ii) this inequality is true.

□

Now two relations are left: $A;BC$ versus $B;CA$ and $C;BA$ versus $B;AC$. By the dashed lines in Figure 7.2.8 on page 61 we signalise that we can not rank these scenarios in the way done until now. Based on the experiments we suppose, fixing α and γ and varying β leads to different relations. In the following, Theorem 7.2.4 confirms this observation.

Theorem 7.2.4 *For all* $\alpha, \beta, \gamma \in (0,1)$ *with* $\alpha < \beta < \gamma$ *are*

(i)

$$P_{A;BC}(3,2,\{1,2\}) \gtreqless P_{B;CA}(3,2,\{1,2\})$$

$$\iff \quad \beta \gtreqless \frac{(1+\alpha) \cdot (\alpha \cdot (1+\gamma) - 1) + (1+\gamma) \cdot (1+\alpha\gamma)}{(1+\alpha) \cdot (1+2\gamma)} \quad \textit{and}$$

(*ii*)

$$P_{B;AC}(3,2,\{1,2\}) \gtreqless P_{C;BA}(3,2,\{1,2\})$$

$$\iff \quad \beta \gtreqless \frac{(1+\gamma)\cdot(\gamma\cdot(1+\alpha)-1)+(1+\alpha)\cdot(1+\alpha\gamma)}{(1+\gamma)\cdot(1+2\alpha)} \quad .$$

Proof. (*i*) $P_{A;BC}(3,2,\{1,2\}) \gtreqless P_{B;CA}(3,2,\{1,2\})$

$$\iff \quad P_A(3,2,\{1,2\})\cdot P_{BC}(3,2,\{1,2\}) \gtreqless P_B(3,2,\{1,2\})\cdot P_{CA}(3,2,\{1,2\})$$

$$\iff \quad \frac{p_2(\alpha)}{p_3(\alpha)}\cdot\frac{p_2(\beta)}{p_3(\beta)}\cdot\frac{p_1(\gamma)+\gamma\,p_2(\gamma)+\beta\cdot[\gamma\,p_1(\gamma)+p_2(\gamma)]}{p_3(\gamma)} \gtreqless$$
$$\frac{p_2(\beta)}{p_3(\beta)}\cdot\frac{p_2(\gamma)}{p_3(\gamma)}\cdot\frac{p_1(\alpha)+\alpha\,p_2(\alpha)+\gamma\cdot[\alpha\,p_1(\alpha)+p_2(\alpha)]}{p_3(\alpha)}$$

$$\iff \quad p_2(\alpha)\cdot(p_1(\gamma)+\gamma\,p_2(\gamma)+\beta\cdot[\gamma\,p_1(\gamma)+p_2(\gamma)]) \gtreqless$$
$$p_2(\gamma)\cdot(p_1(\alpha)+\alpha\,p_2(\alpha)+\gamma\cdot[\alpha\,p_1(\alpha)+p_2(\alpha)])$$

$$\iff \quad (1+\alpha)\cdot(1+\gamma(1+\gamma)+\beta\cdot[\gamma+(1+\gamma)]) \gtreqless$$
$$(1+\gamma)\cdot(1+\alpha(1+\alpha)+\gamma\cdot[\alpha+(1+\alpha)])$$

$$\iff \quad \beta \gtreqless \frac{(1+\alpha)\cdot(\alpha\cdot(1+\gamma)-1)+(1+\gamma)\cdot(1+\alpha\gamma)}{(1+\alpha)\cdot(1+2\gamma)}$$

(*ii*) $P_{B;AC}(3,2,\{1,2\}) \gtreqless P_{C;BA}(3,2,\{1,2\})$

$$\iff \quad P_B(3,2,\{1,2\})\cdot P_{AC}(3,2,\{1,2\}) < P_C(3,2,\{1,2\})\cdot P_{BA}(3,2,\{1,2\})$$

$$\iff \quad \frac{p_2(\beta)}{p_3(\beta)}\cdot\frac{p_2(\alpha)}{p_3(\alpha)}\cdot\frac{p_1(\gamma)+\gamma\,p_2(\gamma)+\alpha\cdot[\gamma\,p_1(\gamma)+p_2(\gamma)]}{p_3(\gamma)} \gtreqless$$
$$\frac{p_2(\gamma)}{p_3(\gamma)}\cdot\frac{p_2(\beta)}{p_3(\beta)}\cdot\frac{p_1(\alpha)+\alpha\,p_2(\alpha)+\beta\cdot[\alpha\,p_1(\alpha)+p_2(\alpha)]}{p_3(\alpha)}$$

$$\iff \quad p_2(\alpha)\cdot(p_1(\gamma)+\gamma\,p_2(\gamma)+\alpha\cdot[\gamma\,p_1(\gamma)+p_2(\gamma)]) \gtreqless$$
$$p_2(\gamma)\cdot(p_1(\alpha)+\alpha\,p_2(\alpha)+\beta\cdot[\alpha\,p_1(\alpha)+p_2(\alpha)])$$

$$\iff \quad (1+\alpha)\cdot(1+\gamma\cdot(1+\gamma)+\alpha\cdot[\gamma+(1+\gamma)]) \gtreqless$$
$$(1+\gamma)\cdot(1+\alpha\cdot(1+\alpha)+\beta\cdot[\alpha+(1+\alpha)])$$

$$\iff \quad \beta \lesseqgtr \frac{(1+\gamma)\cdot(\gamma\cdot(1+\alpha)-1)+(1+\alpha)\cdot(1+\alpha\gamma)}{(1+\gamma)\cdot(1+2\alpha)}$$

□

To illustrate these relations we give an example for fixed α and γ.

Example Consider $\alpha = \frac{1}{8}$ and $\gamma = \frac{7}{8}$. Then

$$\begin{aligned} & P_{A;BC}(3,2,\{1,2\}) \gtreqless P_{B;CA}(3,2,\{1,2\}) \\ \iff \quad & \beta \gtreqless \frac{\left(1+\frac{1}{8}\right)\cdot\left(\frac{1}{8}\cdot\left(1+\frac{7}{8}\right)-1\right)+\left(1+\frac{7}{8}\right)\cdot\left(1+\frac{1}{8}\cdot\frac{7}{8}\right)}{\left(1+\frac{1}{8}\right)\cdot\left(1+2\cdot\frac{7}{8}\right)} \\ \iff \quad & \beta \gtreqless \frac{13}{33} \end{aligned}$$

and

$$\begin{aligned} & P_{B;AC}(3,2,\{1,2\}) \gtreqless P_{C;BA}(3,2,\{1,2\}) \\ \iff \quad & \beta \lesseqgtr \frac{\left(1+\frac{7}{8}\right)\cdot\left(\frac{7}{8}\cdot\left(1+\frac{1}{8}\right)-1\right)+\left(1+\frac{1}{8}\right)\cdot\left(1+\frac{1}{8}\cdot\frac{7}{8}\right)}{\left(1+\frac{7}{8}\right)\cdot\left(1+2\cdot\frac{1}{8}\right)} \\ \iff \quad & \beta \lesseqgtr \frac{13}{25}. \end{aligned}$$

So, there are three intervals with different relations (shortly $P_{A;BC} := P_{A;BC}(3,2,\{1,2\})$):

- $P_{A;BC} < P_{B;CA}$ and $P_{B;AC} > P_{C;BA}$ for $\frac{1}{8} < \beta < \frac{13}{33}$,
- $P_{A;BC} > P_{B;CA}$ and $P_{B;AC} > P_{C;BA}$ for $\frac{13}{33} < \beta < \frac{13}{25}$, and
- $P_{A;BC} > P_{B;CA}$ and $P_{B;AC} < P_{C;BA}$ for $\frac{13}{25} < \beta < \frac{7}{8}$.

Obviously, these intervals are not symmetric. The length of the first interval $\left(\frac{13}{33} - \frac{1}{8} = \frac{71}{264}\right)$ is smaller than the length of the third interval $\left(\frac{7}{8} - \frac{13}{25} = \frac{71}{200}\right)$.

7.3 The Group Selection Problem with Four Experts

The last section of this chapter contains experimental and theoretical results for 4-GSeP. In the first part we show some theoretical results for the scenarios of $(2;2)$-type. After that the second part of this section includes several experimental results for the scenarios of $(1;3)$-type. And the last part of this section presents relationships between scenarios of both types.

As already mentioned, we start with several theoretical results for the scenarios of $(2;2)$-type. Therefore, Figure 7.3.1 on page 69 shows the relations between each four scenarios of groups AB and CD, AC and BD, and also AD and BC. The relations between the three winners ($BA;DC$, $CA;DB$, and $DA;CB$) are drawn in, too.

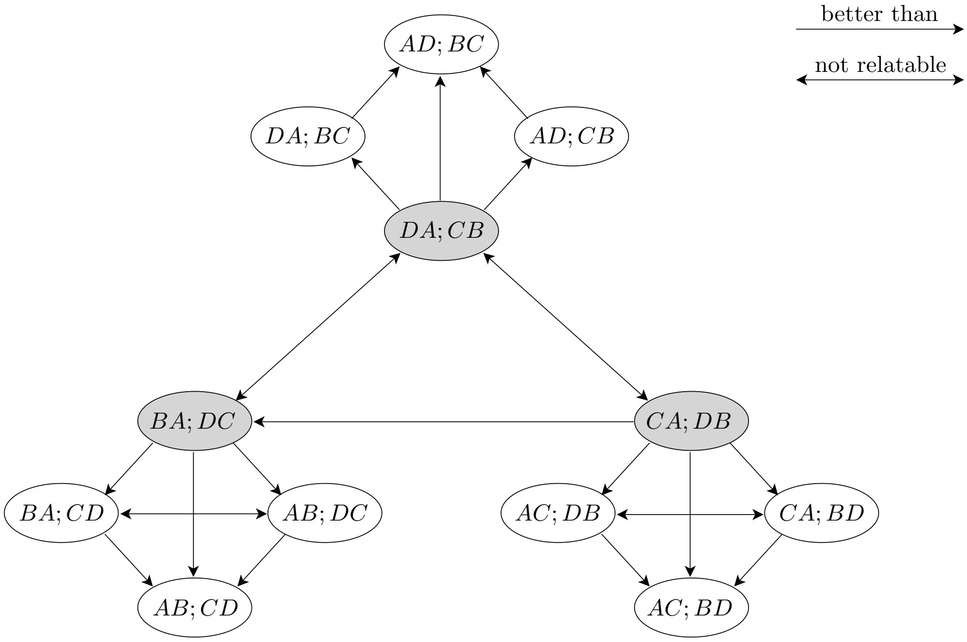


Figure 7.3.1 Chosen relations between all 4-GSeP scenarios of $(2;2)$-type for $n = 3$ and $k = 2$.

There are four dashed lines in Figure 7.3.1. Pairs of scenarios marked with dashed lines are not relatable for all $\alpha, \beta, \gamma, \delta \in (0,1)$ with $\alpha < \beta < \gamma < \delta$. We show these relations using two counterexamples. The results are presented in Table 7.3.1 on page 70.

	$(\alpha,\beta,\gamma,\delta)=\left(\frac{2}{32},\frac{5}{32},\frac{6}{32},\frac{31}{32}\right)$	$(\alpha,\beta,\gamma,\delta)=\left(\frac{2}{32},\frac{27}{32},\frac{28}{32},\frac{31}{32}\right)$
$P_{BA;DC}(3,2,\{1,2\})$	$\frac{6203006517248}{11037535997579}\approx 0.561992$	$\frac{1119644286976}{4364683880115}\approx 0.256524$
	larger than	**smaller than**
$P_{DA;CB}(3,2,\{1,2\})$	$\frac{12026982170624}{21494149047917}\approx 0.559547$	$\frac{31024427827200}{120174296165833}\approx 0.258162$
	smaller than	**larger than**
$P_{CA;DB}(3,2,\{1,2\})$	$\frac{395002773504}{693359646707}\approx 0.569694$	$\frac{4431701606400}{17167756595119}\approx 0.258141$
$P_{AB;DC}(3,2,\{1,2\})$	$\frac{12610829287424}{24022872465319}\approx 0.524951$	$\frac{9077581152256}{45444061575315}\approx 0.199753$
	larger than	**smaller than**
$P_{BA;CD}(3,2,\{1,2\})$	$\frac{50378087333888}{109794437028549}\approx 0.458840$	$\frac{32371571163136}{128321706075381}\approx 0.252269$
$P_{AC;DB}(3,2,\{1,2\})$	$\frac{404666449920}{774931369849}\approx 0.522197$	$\frac{1418144514048}{7069076245049}\approx 0.200612$
	larger than	**smaller than**
$P_{CA;BD}(3,2,\{1,2\})$	$\frac{5603187490816}{12199381892061}\approx 0.459301$	$\frac{4626317312000}{18331672296483}\approx 0.252368$

Table 7.3.1 Counterexamples for pairs of scenarios marked with dashed lines (rounded to six decimals).

After proving the relations marked with dashed lines using counterexamples, we look at the solid lined realtions. All solid lined relations between the four-scenario-groups $(AB);(CD)$, $(AC);(BD)$, and $(AD);(BC)$ are true as a consequence of Theorem 3.4.2 on page 33. Obviously, within these groups the relation between $AD;CB$ and $DA;BC$ remains. In all of our experiments for $n=3$ and $k=2$ we observe $DA;BC$ is better than $AD;CB$. Until now we did not find an example proving this observation for $n=3$ and $k=2$ wrong.

Conjecture 7.3.1 *For all $\alpha,\beta,\gamma,\delta\in(0,1)$ with $\alpha<\beta<\gamma<\delta$ is*

$$P_{DA;BC}(3,2,\{1,2\}) > P_{AD;CB}(3,2,\{1,2\}).$$

Within the winners $BA;DC$, $CA;DB$, and $DA;CB$ of each four-scenario-group we already showed that $DA;CB$ and $BA;DC$ or $DA;CB$ and $CA;DB$ are not relatable. The next theorem shows the remaining relation between $CA;DB$ and $BA;DC$.

Theorem 7.3.1 *For all $\alpha,\beta,\gamma,\delta\in(0,1)$ with $\alpha<\beta<\gamma<\delta$ is*

$$P_{CA;DB}(3,2,\{1,2\}) > P_{BA;DC}(3,2,\{1,2\}).$$

Proof. $P_{CA;DB}(3,2,\{1,2\}) > P_{BA;DC}(3,2,\{1,2\})$

$$\iff P_{CA}(3,2,\{1,2\}) \cdot P_{DB}(3,2,\{1,2\}) > P_{BA}(3,2,\{1,2\}) \cdot P_{DC}(3,2,\{1,2\})$$

$$\iff \frac{p_2(\gamma)}{p_3(\gamma)} \cdot \frac{p_1(\alpha) + \alpha\, p_2(\alpha) + \gamma \cdot [\alpha\, p_1(\alpha) + p_2(\alpha)]}{p_3(\alpha)} \cdot \frac{p_2(\delta)}{p_3(\delta)} \cdot \frac{p_1(\beta) + \beta\, p_2(\beta) + \delta \cdot [\beta\, p_1(\beta) + p_2(\beta)]}{p_3(\beta)} > \frac{p_2(\beta)}{p_3(\beta)} \cdot \frac{p_1(\alpha) + \alpha\, p_2(\alpha) + \beta \cdot [\alpha\, p_1(\alpha) + p_2(\alpha)]}{p_3(\alpha)} \cdot \frac{p_2(\delta)}{p_3(\delta)} \cdot \frac{p_1(\gamma) + \gamma\, p_2(\gamma) + \delta \cdot [\gamma\, p_1(\gamma) + p_2(\gamma)]}{p_3(\gamma)}$$

$$\iff p_2(\gamma) \cdot (p_1(\alpha) + \alpha\, p_2(\alpha) + \gamma \cdot [\alpha\, p_1(\alpha) + p_2(\alpha)]) \cdot (p_1(\beta) + \beta\, p_2(\beta) + \delta \cdot [\beta\, p_1(\beta) + p_2(\beta)]) > p_2(\beta) \cdot (p_1(\alpha) + \alpha\, p_2(\alpha) + \beta \cdot [\alpha\, p_1(\alpha) + p_2(\alpha)]) \cdot (p_1(\gamma) + \gamma\, p_2(\gamma) + \delta \cdot [\gamma\, p_1(\gamma) + p_2(\gamma)])$$

$$\iff (1+\gamma) \cdot (1 + \alpha \cdot (1+\alpha) + \gamma \cdot [\alpha + (1+\alpha)]) \cdot (1 + \beta \cdot (1+\beta) + \delta \cdot [\beta + (1+\beta)]) > (1+\beta) \cdot (1 + \alpha \cdot (1+\alpha) + \beta \cdot [\alpha + (1+\alpha)]) \cdot (1 + \gamma \cdot (1+\gamma) + \delta \cdot [\gamma + (1+\gamma)])$$

$$\iff (\beta - \gamma) \cdot (\alpha^2\beta\gamma + \beta\gamma + \alpha\beta\gamma + \alpha^2(\beta + \gamma + \delta) - [4\alpha\beta\gamma\delta + \beta\delta + \gamma\delta + \alpha \cdot (\beta + \gamma + \delta) + 2\delta \cdot (\alpha\beta + \alpha\gamma + \beta\gamma) + 1 + 2\alpha]) > 0$$

$$\iff \alpha\beta\gamma \cdot (4\delta - \alpha) + \beta \cdot (\delta - \gamma) + \gamma \cdot (\delta - \alpha\beta) + \alpha \cdot (1-\alpha) \cdot (\beta + \gamma + \delta) + 2\delta \cdot (\alpha\beta + \alpha\gamma + \beta\gamma) + 1 + 2\alpha > 0$$

For $\alpha, \beta, \gamma, \delta \in (0,1)$ and $\alpha < \beta < \gamma < \delta$ the following differences are larger than zero:

$$\begin{aligned} 4\delta - \alpha &> 0\,, \\ \delta - \gamma &> 0\,, \\ \delta - \alpha\beta &> 0\,, \text{ and} \\ 1 - \alpha &> 0\,. \end{aligned}$$

So, each summand is larger than zero and consequently the sum of these is larger than zero.

□

The middle part of this section contains several experimental results for the scenarios of $(1,3)$-type. There are 24 of them in total. Therefore, we defined five different arrays of noise levels $v^{(i)} = \left(v_a^{(i)}, v_b^{(i)}, v_c^{(i)}, v_d^{(i)}\right)$ $(i = 1, 2, \ldots, 5)$ for experts A, B, C, and D.

$$
\begin{aligned}
v^{(1)} &= \left(\tfrac{4}{16}, \tfrac{8}{16}, \tfrac{12}{16}, \tfrac{16}{16} \right) = \left(\tfrac{1}{4}, \tfrac{1}{2}, \tfrac{3}{4}, 1 \right) \\
v^{(2)} &= \left(\tfrac{4}{16}, \tfrac{5}{16}, \tfrac{15}{16}, \tfrac{16}{16} \right) = \left(\tfrac{1}{4}, \tfrac{5}{16}, \tfrac{15}{16}, 1 \right) \\
v^{(3)} &= \left(\tfrac{4}{16}, \tfrac{9}{16}, \tfrac{10}{16}, \tfrac{16}{16} \right) = \left(\tfrac{1}{4}, \tfrac{9}{16}, \tfrac{5}{8}, 1 \right) \\
v^{(4)} &= \left(\tfrac{4}{16}, \tfrac{5}{16}, \tfrac{6}{16}, \tfrac{16}{16} \right) = \left(\tfrac{1}{4}, \tfrac{5}{16}, \tfrac{3}{8}, 1 \right) \\
v^{(5)} &= \left(\tfrac{4}{16}, \tfrac{14}{16}, \tfrac{15}{16}, \tfrac{16}{16} \right) = \left(\tfrac{1}{4}, \tfrac{7}{8}, \tfrac{15}{16}, 1 \right)
\end{aligned}
$$

Based on these noise levels we ran each $T = 10^9$ simulations for $n = 4$ and $k = 3$. So, each expert in a triple-expert-group selects exactly one item. Due to the fact that there are $\frac{24 \cdot 23}{2} = 276$ relations between all scenarios, we present an assortment of 46 relations only. The values below the name of a scenario represent the rank of this scenario for the considered noise levels. There are detailed results for these rankings in Table D.1.1 on page 112.

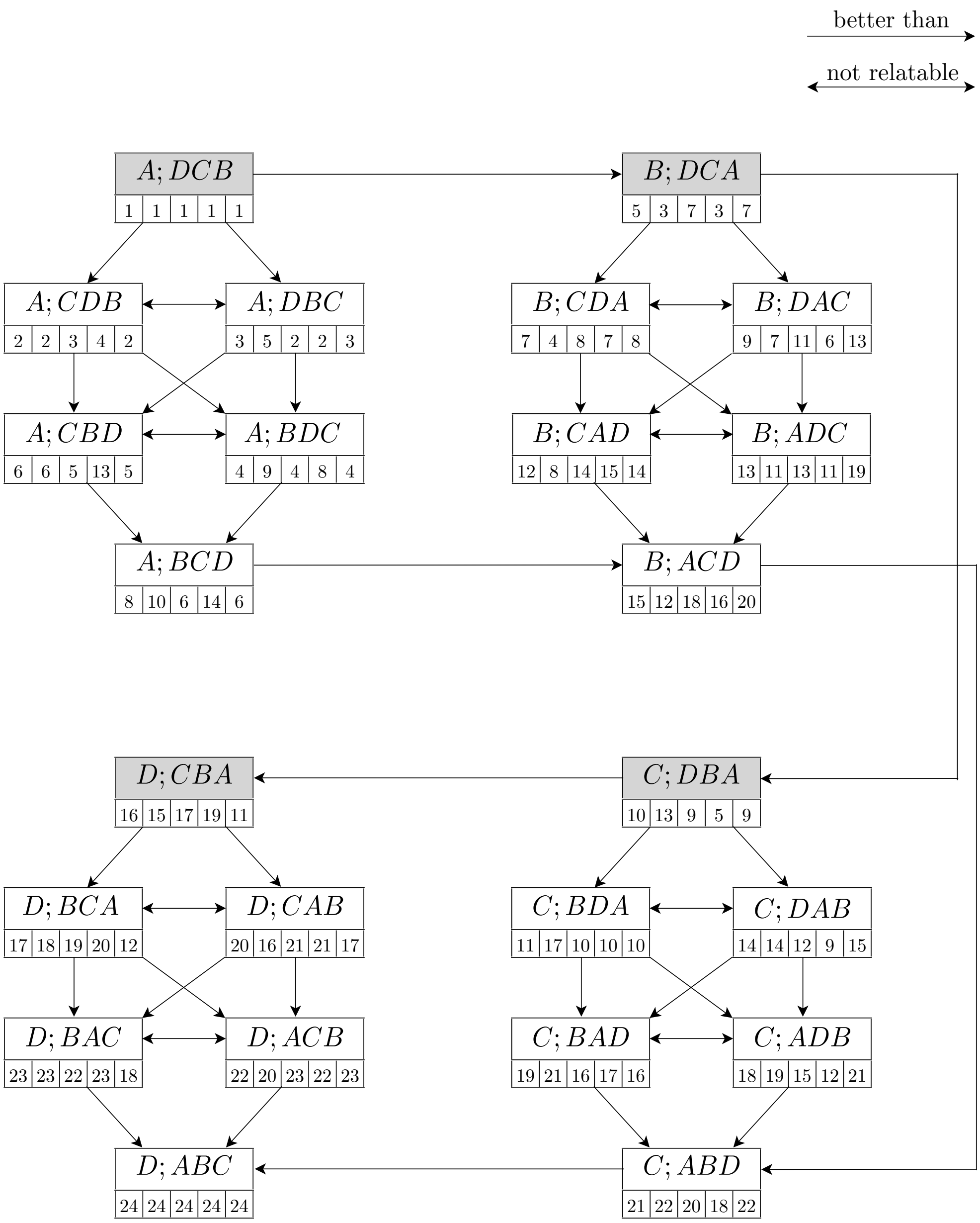


Figure 7.3.2 Chosen, experimental relations between all 4-GSeP scenarios of $(1;3)$-type for $n = 4$, $k = 3$, and $T = 10^9$. The ith value below each scenario represents the rank of this scenario for $v^{(i)}$ $(i = 1, 2, \ldots, 5)$.

With the experimental results for $v^{(i)}$ $(i = 1, 2, \ldots, 5)$ there are several observations within each and beyond all six-scenario-groups. Firstly, we describe the relations within each six-scenario-group by focussing on the triple-expert-groups only. For this purpose, we define $E = \{E_1E_2E_3 : v_{e_1}^{(i)} < v_{e_2}^{(i)} < v_{e_3}^{(i)},\ i = 1, 2, \ldots, 5\} = \{ABC, ABD, ACD, BCD\}$. So, within each six-scenario-group we observe

- $E_3E_2E_1$ is best,
- $E_2E_3E_1$ and $E_3E_1E_2$ are not relatable,
- $E_2E_1E_3$ and $E_1E_3E_2$ are not relatable,

- $E_2E_3E_1$ and $E_3E_1E_2$ are each better than $E_2E_1E_3$ and $E_1E_3E_2$, and
- $E_1E_2E_3$ is worst.

As a consequence, scenarios $A;DCB$, $B;DCA$, $C;DBA$, and $D;CBA$ are the winners within their six-scenario-group. After these results we give an overview of the four main results beyond the six-scenario-groups now:

- $A;DCB$ is better than $B;DCA$ is better than $C;DBA$ is better than $D;CBA$,
- $A;BCD$ is better than $B;ACD$ is better than $C;ABD$ is better than $D;ABC$,
- $A;DCB$ is best, and
- $D;ABC$ is worst.

Beside the relations between the best and the worst scenarios within the six-scenario-groups there are similar relations for the other four scenarios (second or third best and second or third worst). To keep Figure 7.3.2 on page 73 clear, we had to go without these relations in this figure. So, we describe them now.

Second or third best scenarios (upper pair):

- $A;CDB$ is better than $B;CDA$ is better than $C;BDA$ is better than $D;BCA$ and
- $A;DBC$ is better than $B;DAC$ is better than $C;DAB$ is better than $D;CAB$.

Second or third worst scenarios (lower pair):

- $A;CBD$ is better than $B;CAD$ is better than $C;BAD$ is better than $D;BAC$ and
- $A;BDC$ is better than $B;ADC$ is better than $C;ADB$ is better than $D;ACB$.

After presenting all illustrated and several, not illustrated relations in Figure 7.3.2 on page 73 there is one additional, important observation. Although there are consistent relations between the best, second best, ..., and worst scenarios within the six-scenario-groups, a six-scenario-group is not totally better than any other one in general. To give an example, $A;BCD$ is worse than $B;DCA$ and $C;DBA$ for $v^{(4)} = \left(\frac{1}{4}, \frac{5}{16}, \frac{3}{8}, 1\right)$ and also $B;ACD$ is worse than $C;DBA$ and $D;CBA$ for $v^{(5)} = \left(\frac{1}{4}, \frac{7}{8}, \frac{15}{16}, 1\right)$ (see Figure 7.3.2 on page 73). But for these experiments each $A;(BCD)$-scenario (= scenarios of the six-scenario-group including group A) is better than $D;(ABC)$. So, $A;DCB$, ..., and $A;BCD$ each are better than $D;CBA$, ..., and $D;ABC$.

Until now we described the relations between the scenarios of $(2;2)$- and $(1;3)$-type separately. In the last part of this section we focus on the ranking of all scenarios of 4-GSeP. With 12 scenarios of $(2;2)$-type and 24 scenarios of $(1;3)$-type there are 36 of them in total. So, which type of scenarios is best or worst? Answering this question, Table 7.3.2 shows experimental results for all scenarios of 4-GSeP for the noise levels defined on page 72. In this table we only entered the rankings for the different noise levels. Two more detailed tables including the average total sum of each scenario are shown in Appendix D.1 on pages 113 and 114.

scenario	rank $v^{(1)}$	rank $v^{(3)}$	rank $v^{(5)}$
$A;DCB$	1	1	1
$A;CDB$	2	3	2
$A;DBC$	3	2	3
$A;CBD$	4	5	4
$A;BDC$	5	4	5
$A;BCD$	6	6	6
$DA;CB$	7	7	7
$CA;DB$	8	9	9
$DA;BC$	9	8	8
$CA;BD$	10	11	10
$AD;CB$	11	12	13
$AC;DB$	12	15	15
$BA;DC$	13	10	11
$AD;BC$	14	14	14
$BA;CD$	15	13	12
$AC;BD$	16	17	16
$AB;DC$	17	16	17
$AB;CD$	18	18	18
$B;DCA$	19	19	19
$B;CDA$	20	20	20
$B;DAC$	21	21	21
$B;CAD$	22	22	22
$B;ADC$	23	23	23
$B;ACD$	24	24	24
$C;DBA$	25	25	25
$C;BDA$	26	26	26
$C;DAB$	27	27	27
$C;BAD$	28	28	28
$C;ADB$	29	29	29
$C;ABD$	30	30	30
$D;CBA$	31	31	31
$D;BCA$	32	32	32
$D;CAB$	33	33	33
$D;BAC$	34	34	34
$D;ACB$	35	35	35
$D;ABC$	36	36	36

scenario	rank $v^{(2)}$	rank $v^{(4)}$
$A;DCB$	1	1
$A;CDB$	2	3
$A;DBC$	3	2
$A;CBD$	4	5
$A;BDC$	5	4
$A;BCD$	6	6
$B;DCA$	7	7
$B;CDA$	8	9
$B;DAC$	9	8
$B;CAD$	10	11
$B;ADC$	11	10
$B;ACD$	12	12
$DA;CB$	13	13
$CA;DB$	14	15
$DA;BC$	15	14
$CA;BD$	16	20
$AC;DB$	17	16
$AD;CB$	18	19
$AD;BC$	19	21
$AC;BD$	20	23
$BA;DC$	21	17
$BA;CD$	22	22
$AB;DC$	23	18
$AB;CD$	24	24
$C;DBA$	25	25
$C;DAB$	26	26
$C;BDA$	27	27
$C;ADB$	28	28
$C;BAD$	29	29
$C;ABD$	30	30
$D;CBA$	31	31
$D;CAB$	32	33
$D;BCA$	33	32
$D;ACB$	34	35
$D;BAC$	35	34
$D;ABC$	36	36

Table 7.3.2 Ranking of all 36 scenarios of 4-GSeP for $n = 96$, $k = 48$, $v^{(1)} = \left(\frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1\right)$, $v^{(2)} = \left(\frac{1}{4}, \frac{5}{16}, \frac{15}{16}, 1\right)$, $v^{(3)} = \left(\frac{1}{4}, \frac{9}{16}, \frac{5}{8}, 1\right)$, $v^{(4)} = \left(\frac{1}{4}, \frac{5}{16}, \frac{3}{8}, 1\right)$, $v^{(5)} = \left(\frac{1}{4}, \frac{7}{8}, \frac{15}{16}, 1\right)$, and $T = 10^8$ runs.

Obviously, the different types of scenarios are ranked in blocks for all considered noise levels. In each of these experiments scenario $A;DCB$ is best and scenario $D;ABC$ is worst. Whereas the position of the $A;(BCD)$-, $C;(ABD)$-, and $D;(ABC)$-blocks stays constant, the positions of the $(2;2)$- and the $B;(ACD)$-block varies dependent on the noise level. In fact, scenarios of $(2;2)$-type never lead to the best results. Compared with the scenarios of $(1,3)$-type, scenarios of $(2;2)$-type are ranked in the upper half (for $v^{(1)}$, $v^{(3)}$, and $v^{(5)}$) or around the middle (for $v^{(2)}$ and $v^{(4)}$). As a consequence, scenarios of $(1;3)$-type always lead to the best and worst results. So, the choice of a scenario of this type is of very high importance.

8 Conclusion and Discussion

In this thesis we investigated altogether six models of characteristic selection problems. This chapter summarises the main results and observations in these models. It also discusses several interesting structures that appeared.

8.1 On Differences between the Characterised Selection and Deletion Scenarios of 2-SeP

For the Selection Problem with two experts we analysed two different cases: experts with equal and experts with different noise levels.

For the first case the main result is that each specified double-expert-scenario consisting of selecting or deleting actions is at least as good as or totally better than the single-expert-scenario(s). Especially, both the pure selection scenario AB and the pure deletion scenario ab are better than scenario A (or a, B, and b). The single mixed scenarios Ab and aB are also better than scenario A for medium selection numbers k. But, for small and large k both of them are equally good as scenario A. The reason for this effect is the slight influence of the deleting expert. Considering scenario Ab for small and large k, expert B deletes a little number of the worst items until the scenario stops. So, expert B usually not deletes an item expert A wants to select. Consequently, selecting a small or large number of items by using the single mixed scenarios Ab or aB is nearly the same as letting A or B select these items without the influence of the other expert. Allowing selection and deletion actions and equal number of A- and B-actions, so as in the double mixed scenarios $ABab$ and $abAB$, totally improves the result of the single mixed scenarios Ab or aB. So, in our experiments we achieved the best results for the pure selection scenario AB (large k), the pure deletion scenario ab (small k), and the double mixed scenarios $ABab$ and $abAB$ (medium size k).

Among the pure, single mixed, and double mixed scenarios there also appeared a remarkable effect: For $0 < k < \frac{n}{2}$ scenarios starting with a deletion action are better than their complementary scenarios starting with a selection action (e.g. ab is better than AB). And for $\frac{n}{2} < k < n$ it is the other way around. As a reason for this effect we assume the number of crucial actions for this scenarios an intervals. For all considered scenarios the scenario with larger number of crucial actions is better than its complementary scenario. So, for $0 < k < \frac{n}{2}$ ab is better than AB, aB is better than Ab, and $abAB$ is better than $ABab$ because the number of crucial actions is always larger. Table 8.1.1 gives a short overview of the numbers of crucial actions for all considered scenarios.

scenarios	# crucial actions	
	$0 < k < \frac{n}{2}$	$\frac{n}{2} < k < n$
AB versus ab	$k < n - k$	$k > n - k$
Ab versus aB	$2k - 1 < 2k$	$2(n-k) > 2(n-k) - 1$
$ABab$ versus $abAB$	$4k - 2 < 4k$	$2(n-k) > 2(n-k) - 2$

Table 8.1.1 Number of crucial actions in the complementary scenarios.

For $0 < k < \frac{n}{2}$ the number of crucial actions of scenarios starting with a deletion action is at least one times larger than those of scenarios starting with a selection action. The absolute differences between each two numbers of crucial actions according to their amount are $n - 2k$ (even k with $0 < k < n$) for AB and ab, 1 for Ab and aB, and 2 for $ABab$ and $abAB$.

If experts A and B have different noise levels (B is worse than A in general), there are also two interesting observations.

Firstly, the double-expert-scenarios **can** improve the result of the single-expert-scenarios although B is worse than A. In our experiments we investigated all characterised double-expert-scenarios for several combinations of noise levels for experts A and B. It came out that even if noise level v_b is twice as large as v_a (case: $v_a = \frac{1}{4}$ and $v_b = \frac{1}{2}$), double-expert-scenarios are better than single-expert-scenarios for several selection numbers k. But, for this case none of the double-expert-scenarios is better than scenario A for all possible k.

The second main observation for 2-SeP with different noise levels is that B-starting pure s[...] verse scenar[...]

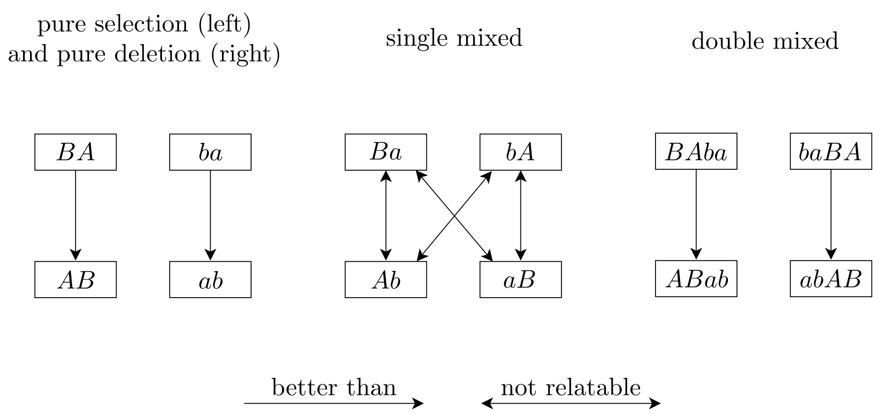


Figure 8.1.1 Relations between reverse double-expert-scenarios for experts with different noise levels (B is worse than A).

In contrast to the relations between the four left and four right scenarios in this figure, the single mixed scenarios Ab, Ba, bA, and aB are not relatable in this way. Comparing the structure of all these types of scenarios gives the reason for this contrast. Reversing the positions of experts A and B in the pure selection, pure deletion and double mixed scenarios does not reverse the tasks of experts A or B. For example, by reversing the positions of A and B in scenario AB both experts still have the task to **select** a given number of items (A and B are "selecting" actors). In contrast to this, reversing the positions of A and B in the single mixed scenarios Ab and Ba also changes the tasks of both experts:

Scenario Ab:
A is a **starting** and **selecting** actor and B is a **non-starting** and **deleting** actor.

Scenario Ba:
A is a **non-starting** and **deleting** actor and B is a **starting** and **selecting** actor.

There is no clear relation between Ba and aB or Ab and bA, too. Here the task of each expert remains the same, but in contrast to pure selection, pure deletion and double mixed scenarios we would compare scenarios starting with a selection action with those starting with a deletion action.

8.2 On Rankings of Pure Selection Scenarios of 2-SeP

After analysing scenario AB we widened our investigations of pure selection scenarios. For this purpose, we loosened the restriction of strictly alternating actions of experts A and B and also allowed different numbers of action of both experts. This gave us the opportunity to consider $\frac{1}{2} \cdot 2^k$ (experts with equal noise levels) or 2^k (experts with different noise levels) selection orders with $\#A$ actions + $\#B$ actions $= k$.

For equally good experts we observed the following results and structures in the rankings of pure selection scenarios:

- By increasing the number of B-actions from 0 to $\lfloor \frac{k}{2} \rfloor$
  - the performance **always** increases. Consequently, the best scenarios are **always** of $A^{\lceil \frac{k}{2} \rceil} B^{\lfloor \frac{k}{2} \rfloor}$-type ($\#A = \lceil \frac{k}{2} \rceil$ and $\#B = \lfloor \frac{k}{2} \rfloor$). But within these scenarios, AB is **never** best. Further, scenario A is **always** worst.
  - the absolute performance difference between the best $A^{k-i}B^i$- and the best $A^{k-(i+1)}B^{i+1}$-scenario ($i = 0, \ldots, \lfloor \frac{k}{2} \rfloor - 1$) **always** decreases.
- Within the $A^{k-1}B^1$-scenarios there is **always**

  $A \ldots AAB$ best,

  $A \ldots ABA$ second best, ...,

  $ABA \ldots A$ second worst, and

  $BAA \ldots A$ worst.

Beside these results we observed an interesting structure among the A^iB^j-scenarios ($i + j = k$ and $i \geq j$). By increasing n and holding k and $v_a = v_b$ constant the rankings of the A^iB^j-scenarios form blocks. So, for large n and small k the best block consists of $A^{\lceil \frac{k}{2} \rceil} B^{\lfloor \frac{k}{2} \rfloor}$-scenarios, the second best block consists of $A^{\lceil \frac{k}{2} \rceil + 1} B^{\lfloor \frac{k}{2} \rfloor - 1}$-scenarios, ..., and the worst block consists of one $A^k B^0$-scenario (i.e., scenario A). This structure also occurs by increasing $v_a = v_b$ and holding n and k constant.

For experts with different noise levels we observed these results and structures:

- By increasing the number of B-actions from 0 to $\lfloor \frac{k}{2} \rfloor$ the performance **mostly** increases. So, scenario A is **not always** best and scenario B is **always** worst. Further, each $A^{k-1}B^1$-scenario is **better** than each A^1B^{k-1}-scenario. And $A^{\lceil \frac{k}{2} \rceil} B^{\lfloor \frac{k}{2} \rfloor}$-scenarios locate around the middle of each ranking, however B-starting scenarios are **better** than their reverse A-starting scenarios **mostly**.
- Within the A^1B^{k-1}-scenarios there is **always**

  $B \ldots BBA$ best,

  $B \ldots BAB$ second best, ..., and

  $BAB \ldots B$ second worst,

  $ABB \ldots B$ worst.
- Within the $A^{k-1}B^1$-scenarios there is **mostly**

  $BAA \ldots A$ best,

  $ABA \ldots A$ second best, ..., and

  $A \ldots ABA$ second worst,

  $A \ldots AAB$ worst.

Regarding the first bullet listed, scenario A is not always the best scenario, although expert B is worse than expert A. We already noticed this observation for pure deletion and mixed scenarios as summarised in the previous section. By increasing v_b and holding n, k, and v_a constant the block structure as explained for equally good experts also occurs for experts with different noise levels. For small v_a and large v_b the blocks are ranked according to decreasing number of A-actions. So, the A^kB^0-block is best and the A^0B^k-block is worst.

8.3 On Structures of Selection Orders for all Selection Problems

Considering selection orders for all introduced selection problems, except of TeSeP, we observed a uniform structure for the sequence of action for experts with different noise levels. To generalise these sequences we consider $m \leq k$ experts $E_1, E_2, \ldots, E_m$ with noise levels $v_{e_1} < v_{e_2} < \ldots < v_{e_m}$. With these agreements

$$E_m E_{m-1} \ldots E_1 \text{ is better than } E_1 E_2 \ldots E_m.$$

To give an example for $m = 2$ (2-SeP), selection order BA is better than AB. Especially for k-SeP, selection order $E_k E_{k-1} \ldots E_1$ is best and selection order $E_1 E_2 \ldots E_k$

is worst independent of k and the distribution for the noise levels. For the group selection problems, which consist of two independent selection processes, we have these structure for both processes. So, for 4-GSeP we exemplarily have $DA; CB > AD; CB$ ($DA; CB$ is better than $AD; CB$) and particularly $DA; CB > AD; BC$. And within the $A; (BCD)$-scenarios $A; DCB$ is best and $A; BCD$ is worst. Several of the experimental results for GSeP with three and four experts are proved theoretically.

In contrast to the structure described previously, we observed a reverse structure for TeSeP. Here, we consider a competition between two teams AB and C. Team AB more often wins against team C by setting A as the starting actor within its team. So, for winning the competition it is important for team AB trying to select one of the best possible items in the first step. Otherwise the first turn of A is in the third step and C could select a truly good item A wanted to select before.

Altogether, considering two or more experts with different noise levels we observed different structures for best and worst selection orders. For 2-SeP, k-SeP, 3-GSeP, and 4-GSeP we achieved the best results regarding strictly alternating selection orders by letting the worst expert act firstly, the second worst expert secondly, ..., and the best expert finally. Considering a competitive situation as defined in TeSeP, the best selection order is reverse. So, there is no general conclusion answering the question of which selection orders are best or worst. Indeed, the answer to this question depends on the definition of each selection problem or an optimisation problem in general.

9 Open Problems and Outlook

We finish the thesis by giving several approaches, ideas and open questions for future work.

1. In Section 4.1 starting on page 39 we presented rankings for pure selection scenarios for equally good experts. It turned out that the absolute performance differences between the best $A^{k-i}B^{i}$- and the best $A^{k-(i+1)}B^{i+1}$-scenario ($i = 0, \ldots, \lfloor \frac{k}{2} \rfloor - 1$) decreases by increasing the number of B-actions from 0 to $\lfloor \frac{k}{2} \rfloor$. Until now we do not have a good explanation for this effect. So, why does this effect occurs? We also observed that scenario AB never resulted in the best performance (see Table 4.1.4 on page 42). To give an example, for $k = 6$ there are at least three better selection orders. How many better selection orders are there for $n \to \infty$?

2. Regarding the rankings for pure selection scenarios for equally good experts in Section 4.1 starting on page 39 again, the rankings of pure deletion scenarios might result in interesting structures (e.g., $\#a = n - k - 1$ and $\#b = 1$), too. As an extension of the single mixed scenarios Ab and aB or the double mixed scenarios $ABab$ and $abAB$, analysing all possible mixed orders to select k out of n items gives another possible approach for future work:
   (i) single mixed: $(\#A, \#b) = (k, n-k)$ and $(\#B, \#a) = (k, n-k)$
   (ii) double mixed: $(\#A + \#B, \#a + \#b) = (k, n-k)$ or only
   $$(\#A, \#B, \#a, \#b) = (\lceil \tfrac{k}{2} \rceil, \lfloor \tfrac{k}{2} \rfloor, \lceil \tfrac{n-k}{2} \rceil, \lfloor \tfrac{n-k}{2} \rfloor)$$
   Especially, where are scenarios $AbaB$ or $aBAb$ located in comparison to the double mixed scenarios $ABab$ and $abAB$?

3. We defined scenario AB as an alternating sequence of action of experts A and B (see Section 2.1 on page 17). Another possibility to define a pure selection scenario is to include the results of scenarios A and B. The total sum of k out of n selected items regarding the scenarios $i = A, B, AB$ is given by $w_k^i = \sum_{j \in S_k} x_j$.
   (i) $w_k^{AB_{AM}} = \frac{1}{2} \cdot \left(w_k^A + w_k^B \right)$ (arithmetic mean),
   (ii) $w_k^{AB_{min}} = \min\{w_k^A, w_k^B\}$ (minimum function),
   (iii) $w_k^{AB_{max}} = \max\{w_k^A, w_k^B\}$ (maximum function), ...
   How good or bad are these scenarios in comparison to scenario AB and the other scenarios introduced?

4. In their diploma theses, Kupfer ([Kae2010]) and Hilbert ([Hil2010]) investigated models of independent and correlated experts (see Section 3.2 on page 27). Kupfer analysed optimisation problems of sum type and the results with correlated experts were totally contrary. Hilbert investigated several selection methods and for all these problems the results with negatively correlated experts were better than those with positively correlated experts. Considering also models of correlated experts for the selection problems introduced in this thesis, which model performs best? Are there differences between several types of selection problems? Especially regarding the structures of selection orders, we observed contrary structures for 2-Sep, k-SeP, 3-GSeP, and 4-GSeP and for TeSeP.

5. For TeSeP we allowed alternately single actions of each team. For example, the selection order of scenario $AB:C$ is $ACBC$. With this restriction the best order within the double-expert-team is contrary to those for all other types of selection problems with two or more experts introduced. Allowing alternately double actions of each team, is scenario $AB:C$ (selection order $ABCC$) still better than scenario $BA:C$ (selection order $BACC$)?

6. Kolassa analysed models for multi-step shortlisting ([Kol2004a], [Kol2004b]) with imperfect experts. The true values for the items (here: x_i) and the noise values for the experts (here: a_i, b_i, c_i and d_i) were uniformly distributed. Further, he investigated two more distributions: normally and exponentially distributed true and noise values. Varying the distribution of the true or noise values resulted in widely different structures. Considering different distributions for the selection problems, are there performance differences, for example various rankings or structures of (pure) selection orders?

7. For 3-GSeP and 4-GSeP we required independent and non-perfect experts with different noise levels (see Section 2.4 starting on page 22). There are several approaches to drop the independence (i) and difference (ii) between the experts:
   (i) Pairwise exchange of information for $p, q \in [0,1]$ (taken from [Kae2010]):

	A observes	B observes
• single-sided information for A	$x_i + pa_i + (1-p)b_i$	$x_i + b_i$
• single-sided information for B	$x_i + a_i$	$x_i + qb_i + (1-q)a_i$
• double-sided information	$x_i + pa_i + (1-p)b_i$	$x_i + qb_i + (1-q)a_i$

   Holding the restriction $v_a < v_b < v_c < v_d$, are scenarios $A;CB$ (3-GSeP) or $A;DCB$ (4-GSeP) still best?

   (ii) Pairwise, equally good experts, for example $v_a = v_b < v_c = v_d$ for 4-GSeP. Considering this restriction, which scenario is better: $AB;CD$ or $AC;BD$ (or $CA;DB$)?

8. By analogy with the pure deletion scenarios of 2-SeP, scenarios of TeSeP and GSeP can be defined as $ab:c$, $a;bc$, $ab;cd$, and so on. Are there similar structures for these scenarios as there are for pure selection scenarios?

The thesis mainly focusses on extensively experimental investigations of 2-SeP and theoretical observations of 3- and 4-GSeP. To more understand the structures of the rankings of pure selection scenarios the investigation of various distributions for the true and noise values as explained in item 6 might be helpful. Understanding these structures provides an opportunity of proving several experimental observations. Until now we proved several observations of 3- and 4-GSeP for $n = 3$ and $k = 2$. In future, proving these observations for $n > 3$ and $k = 2$ offers a challenging task. As already suggested, varying the distributions for the true and noise values and modifying the definitions for the selection problems and their scenarios will also give lots of potential approaches for future work.

A The Selection Problem with Two Experts

A.1 Experts with Equal Noise Levels

In addition to Section 4.1 starting on page 35, the following figures present the absolute performance differences between the complementary single and double mixed scenarios.

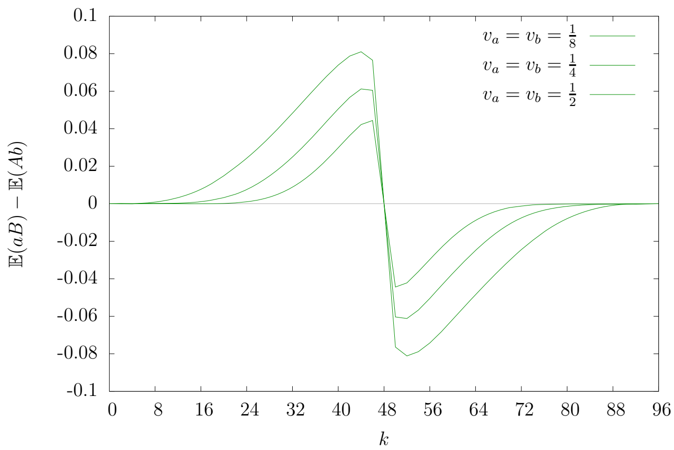


Figure A.1.1 Absolute performance differences between the complementary scenarios aB and Ab for $n = 96$, and $T = 10^8$ runs.

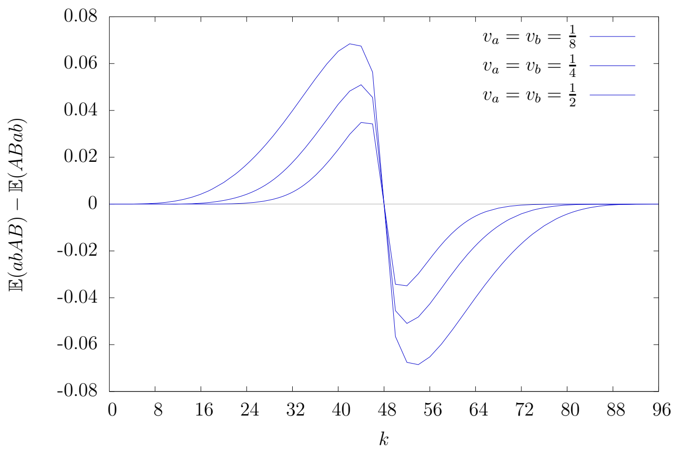


Figure A.1.2 Absolute performance differences between the complementary scenarios $abAB$ and $ABab$ for $n = 96$, and $T = 10^8$ runs.

In Section 4.1 starting on page 37 we analysed the numbers of crucial selection and deletion actions of experts A and B for all considered scenarios. The table below shows how to calculate these numbers for even n and even k in general.

scenario	expert	# selection actions	# deletion actions
A	A	k	-
AB	A	$\frac{k}{2}$	-
	B	$\frac{k}{2}$	-
ab	A	-	$\frac{n-k}{2}$
	B	-	$\frac{n-k}{2}$
Ab	A	$\min\{k, n-k\}$	-
	B	-	$\min\{k-1, n-k\}$
aB	A	-	$\min\{k, n-k-1\}$
	B	$\min\{k, n-k\}$	-
$ABab$	A	$\min\{k, \frac{n-k}{2}\}$	$\min\{k-1, \frac{n-k}{2}\}$
	B	$\min\{k, \frac{n-k}{2}\}$	$\min\{k-1, \frac{n-k}{2}\}$
$abAB$	A	$\min\{k, \frac{n-k}{2}-1\}$	$\min\{k, \frac{n-k}{2}\}$
	B	$\min\{k, \frac{n-k}{2}-1\}$	$\min\{k, \frac{n-k}{2}\}$

Table A.1.1 Number of crucial selection and deletion actions of A and B in the complementary scenarios for even n and even k with $0 < k < n$.

The following pages comprehend detailed rankings of pure selection scenarios for diverse n and $k \in \{3, 5, 6\}$ as described in Section 4.1 on page 42. Scenario AB is highlighted in each table. The average values are rounded to four (sometimes five) decimals always.

#A	#B	selection order			$\sum_{i \in S_3} x_i$
2	1	A	A	B	5.9839
2	1	**A**	**B**	**A**	5.9761
2	1	B	A	A	5.9544
3	0	A	A	A	5.8616

Table A.1.2 Ranking of pure selection scenarios for $n = 96$, $k = 3$, $v_a = v_b = \frac{1}{4}$, and $T = 10^9$ runs.

#A	#B	selection order					$\sum_{i \in S_5} x_i$
3	2	A	A	B	A	B	9.1651
3	2	A	A	B	B	A	9.1634
3	2	A	B	A	A	B	9.1633
3	2	A	A	A	B	B	9.1631
3	2	**A**	**B**	**A**	**B**	**A**	9.1592
3	2	B	A	A	A	B	9.1562
3	2	A	B	B	A	A	9.1514
3	2	B	A	A	B	A	9.1493
3	2	B	A	B	A	A	9.1387
4	1	A	A	A	A	B	9.1285
3	2	B	B	A	A	A	9.1227
4	1	A	A	A	B	A	9.1145
4	1	A	A	B	A	A	9.0965
4	1	A	B	A	A	A	9.0731
4	1	B	A	A	A	A	9.0397
5	0	A	A	A	A	A	8.9480

Table A.1.3 Ranking of pure selection scenarios for $n = 96$, $k = 5$, $v_a = v_b = \frac{1}{4}$, and $T = 10^9$ runs.

#A	#B	selection order	$\sum_{i \in S_6} x_i$
3	3	A B A B B A	4.1249
4	2	A A B A A B	4.1244
3	3	A B B A A B	4.1243
4	2	A A A B A B	4.1241
3	3	A A B B B A	4.1237
3	3	A B B A B A	4.1233
4	2	A A A B B A	4.1228
4	2	A B A A A B	4.1223
3	3	**A B A B A B**	4.1213
4	2	B A A A A B	4.1194
4	2	A A B A B A	4.1183
4	2	A A A A B B	4.1171
3	3	A A B B A B	4.1166
5	1	A A A A A B	4.1163
3	3	A B B B A A	4.1147
4	2	A B A A B A	4.1125
3	3	A B A A B B	4.1109
4	2	B A A A B A	4.1069
4	2	A A B B A A	4.1060
3	3	A A B A B B	4.1035
5	1	A A A A B A	4.1011
4	2	A B A B A A	4.0975
4	2	B A A B A A	4.0895
3	3	A A A B B B	4.0894
4	2	A B B A A A	4.0821
5	1	A A A B A A	4.0813
4	2	B A B A A A	4.0723
5	1	A A B A A A	4.0619
4	2	B B A A A A	4.0566
5	1	A B A A A A	4.0447
5	1	B A A A A A	4.0300
6	0	A A A A A A	4.0156

Table A.1.4 Ranking of pure selection scenarios for $n = 12$, $k = 6$, $v_a = v_b = \frac{1}{4}$, and $T = 10^9$ runs.

#A	#B	selection order	$\sum_{i \in S_6} x_i$
3	3	A B A B B A	6.7419
3	3	A B B A A B	6.7416
3	3	A B B A B A	6.7413
4	2	A A A B A B	6.7410
3	3	A A B B B A	6.7393
4	2	A A B A A B	6.7389
3	3	**A B A B A B**	6.7389
4	2	A A A A B B	6.7385
4	2	A A A B B A	6.7378
3	3	A B B B A A	6.7359
4	2	A B A A A B	6.7340
3	3	A A B B A B	6.7333
4	2	A A B A B A	6.7323
3	3	A B A A B B	6.7312
4	2	B A A A A B	6.7269
4	2	A B A A B A	6.7242
3	3	A A B A B B	6.7229
4	2	A A B B A A	6.7213
5	1	A A A A A B	6.7156
4	2	B A A A B A	6.7141
4	2	A B A B A A	6.7105
3	3	A A A B B B	6.7092
5	1	A A A A B A	6.6985
4	2	B A A B A A	6.6975
4	2	A B B A A A	6.6942
4	2	B A B A A A	6.6788
5	1	A A A B A A	6.6775
4	2	B B A A A A	6.6580
5	1	A A B A A A	6.6546
5	1	A B A A A A	6.6304
5	1	B A A A A A	6.6044
6	0	A A A A A A	6.5664

Table A.1.5 Ranking of pure selection scenarios for $n = 24$, $k = 6$, $v_a = v_b = \frac{1}{4}$, and $T = 10^9$ runs.

#A	#B	selection order	$\sum_{i\in S_6} x_i$
3	3	A B A B B A	8.8308
3	3	A B B A B A	8.83072
3	3	A B B A A B	8.83065
3	3	**A B A B A B**	8.8285
4	2	A A A A B B	8.8284
4	2	A A A B A B	8.8282
3	3	A A B B B A	8.8277
3	3	A B B B A A	8.8273
4	2	A A B A A B	8.8246
4	2	A A A B B A	8.8242
3	3	A A B B A B	8.8232
3	3	A B A A B B	8.8229
4	2	A A B A B A	8.8183
4	2	A B A A A B	8.8181
3	3	A A B A B B	8.8154
4	2	A B A A B A	8.8093
4	2	A A B B A A	8.8088
4	2	B A A A A B	8.8081
3	3	A A A B B B	8.8041
4	2	A B A B A A	8.7976
4	2	B A A A B A	8.7967
5	1	A A A A A B	8.7866
4	2	A B B A A A	8.7829
4	2	B A A B A A	8.7824
5	1	A A A A B A	8.7701
4	2	B A B A A A	8.7652
5	1	A A A B A A	8.7504
4	2	B B A A A A	8.7442
5	1	A A B A A A	8.7278
5	1	A B A A A A	8.7019
5	1	B A A A A A	8.6701
6	0	A A A A A A	8.6072

Table A.1.6 Ranking of pure selection scenarios for $n = 48$, $k = 6$, $v_a = v_b = \frac{1}{4}$, and $T = 10^9$ runs.

#A	#B	selection order	$\sum_{i\in S_6} x_i$
3	3	A B B A B A	10.6141
3	3	A B A B B A	10.61388
3	3	A B B A A B	10.61387
3	3	**A B A B A B**	10.6122
3	3	A B B B A A	10.6118
3	3	A A B B B A	10.6109
4	2	A A A A B B	10.6104
4	2	A A A B A B	10.6087
3	3	A B A A B B	10.6080
3	3	A A B B A B	10.6075
4	2	A A A B B A	10.6046
4	2	A A B A A B	10.6044
3	3	A A B A B B	10.6017
4	2	A A B A B A	10.5985
4	2	A B A A A B	10.5972
3	3	A A A B B B	10.5926
4	2	A A B B A A	10.5903
4	2	A B A A B A	10.5895
4	2	B A A A A B	10.5856
4	2	A B A B A A	10.5795
4	2	B A A A B A	10.5757
4	2	A B B A A A	10.5667
4	2	B A A B A A	10.5635
5	1	A A A A A B	10.5536
4	2	B A B A A A	10.5484
5	1	A A A A B A	10.5383
4	2	B B A A A A	10.5286
5	1	A A A B A A	10.5202
5	1	A A B A A A	10.4990
5	1	A B A A A A	10.4735
5	1	B A A A A A	10.4394
6	0	A A A A A A	10.3530

Table A.1.7 Ranking of pure selection scenarios for $n = 96$, $k = 6$, $v_a = v_b = \frac{1}{4}$, and $T = 10^9$ runs.

#A	#B	selection order	$\sum_{i \in S_6} x_i$
3	3	A B B A B A	12.1908
3	3	A B B A A B	12.1906
3	3	A B A B B A	12.1905
3	3	A B B B A A	12.18929
3	3	**A B A B A B**	12.18926
3	3	A A B B B A	12.1878
3	3	A B A A B B	12.1862
3	3	A A B B A B	12.1852
4	2	A A A A B B	12.1851
4	2	A A A B A B	12.1825
3	3	A A B A B B	12.1808
4	2	A A A B B A	12.1784
4	2	A A B A A B	12.1778
3	3	A A A B B B	12.1736
4	2	A A B A B A	12.1725
4	2	A B A A A B	12.1704
4	2	A A B B A A	12.1653
4	2	A B A A B A	12.1638
4	2	B A A A A B	12.1581
4	2	A B A B A A	12.1552
4	2	B A A A B A	12.1496
4	2	A B B A A A	12.1439
4	2	B A A B A A	12.1391
4	2	B A B A A A	12.1260
5	1	A A A A A B	12.1159
4	2	B B A A A A	12.1080
5	1	A A A A B A	12.1019
5	1	A A A B A A	12.0855
5	1	A A B A A A	12.0660
5	1	A B A A A A	12.0418
5	1	B A A A A A	12.0074
6	0	A A A A A A	11.8996

Table A.1.8 Ranking of pure selection scenarios for $n = 192$, $k = 6$, $v_a = v_b = \frac{1}{4}$, and $T = 10^9$ runs.

#A	#B	selection order	$\sum_{i \in S_6} x_i$
3	3	A B B A B A	13.6166
3	3	A B B A A B	13.6164
3	3	A B A B B A	13.6163
3	3	A B B B A A	13.6155
3	3	**A B A B A B**	13.6153
3	3	A A B B B A	13.6138
3	3	A B A A B B	13.6129
3	3	A A B B A B	13.6119
3	3	A A B A B B	13.6085
4	2	A A A A B B	13.6083
4	2	A A A B A B	13.6054
3	3	A A A B B B	13.6027
4	2	A A A B B A	13.6014
4	2	A A B A A B	13.6006
4	2	A A B A B A	13.5958
4	2	A B A A A B	13.5933
4	2	A A B B A A	13.5894
4	2	A B A A B A	13.5874
4	2	B A A A A B	13.5807
4	2	A B A B A A	13.5800
4	2	B A A A B A	13.5734
4	2	A B B A A A	13.5702
4	2	B A A B A A	13.5645
4	2	B A B A A A	13.5531
4	2	B B A A A A	13.5369
5	1	A A A A A B	13.5290
5	1	A A A A B A	13.5164
5	1	A A A B A A	13.5016
5	1	A A B A A A	13.4838
5	1	A B A A A A	13.4612
5	1	B A A A A A	13.4276
6	0	A A A A A A	13.3006

Table A.1.9 Ranking of pure selection scenarios for $n = 384$, $k = 6$, $v_a = v_b = \frac{1}{4}$, and $T = 10^9$ runs.

After presenting rankings of pure selection scenarios for diverse n, $k \in \{3, 5, 6\}$, and $v_a = v_b = \frac{1}{4}$, we now focus on $n = 96$ and $k = 6$ for different noise levels $v_a = v_b \in \{\frac{1}{8}, \frac{1}{2}, 1\}$. The results are summarised in Table A.1.13 on page 95.

#A	#B	selection order	$\sum_{i \in S_6} x_i$
3	3	A B A B B A	11.0650
3	3	A B B A A B	11.0648
3	3	A B B A B A	11.0647
4	2	A A A B A B	11.0640
3	3	**A B A B A B**	11.0627
4	2	A A A A B B	11.0623
3	3	A A B B B A	11.0622
4	2	A A B A A B	11.0621
4	2	A A A B B A	11.0617
3	3	A B B B A A	11.0608
3	3	A A B B A B	11.0576
4	2	A B A A A B	11.05728
4	2	A A B A B A	11.05727
3	3	A B A A B B	11.0568
4	2	B A A A A B	11.05012
4	2	A B A A B A	11.05008
3	3	A A B A B B	11.0496
4	2	A A B B A A	11.0490
4	2	B A A A B A	11.0404
5	1	A A A A A B	11.0399
4	2	A B A B A A	11.0395
3	3	A A A B B B	11.0380
4	2	B A A B A A	11.0274
5	1	A A A A B A	11.0270
4	2	A B B A A A	11.0260
4	2	B A B A A A	11.0116
5	1	A A A B A A	11.0107
4	2	B B A A A A	10.9928
5	1	A A B A A A	10.9915
5	1	A B A A A A	10.9698
5	1	B A A A A A	10.9451
6	0	A A A A A A	10.9130

Table A.1.10 Ranking of pure selection scenarios for $n = 96$, $k = 6$, $v_a = v_b = \frac{1}{8}$, and $T = 10^9$ runs.

#A	#B	selection order	$\sum_{i \in S_6} x_i$
3	3	A B B A B A	9.8458
3	3	A B B A A B	9.84561
3	3	A B A B B A	9.84557
3	3	A B B B A A	9.8447
3	3	**A B A B A B**	9.8446
3	3	A A B B B A	9.8435
3	3	A B A A B B	9.8424
3	3	A A B B A B	9.8416
3	3	A A B A B B	9.8385
4	2	A A A A B B	9.8335
3	3	A A A B B B	9.8332
4	2	A A A B A B	9.8295
4	2	A A A B B A	9.8246
4	2	A A B A A B	9.8239
4	2	A A B A B A	9.8181
4	2	A B A A A B	9.8160
4	2	A A B B A A	9.8110
4	2	A B A A B A	9.8092
4	2	B A A A A B	9.8035
4	2	A B A B A A	9.8011
4	2	B A A A B A	9.7954
4	2	A B B A A A	9.7909
4	2	B A A B A A	9.7859
4	2	B A B A A A	9.7743
4	2	B B A A A A	9.7586
5	1	A A A A A B	9.7318
5	1	A A A A B A	9.7170
5	1	A A A B A A	9.7003
5	1	A A B A A A	9.6811
5	1	A B A A A A	9.6579
5	1	B A A A A A	9.6250
6	0	A A A A A A	9.4510

Table A.1.11 Ranking of pure selection scenarios for $n = 96$, $k = 6$, $v_a = v_b = \frac{1}{2}$, and $T = 10^9$ runs.

#A	#B	selection order	$\sum_{i \in S_6} x_i$
3	3	A B B A B A	8.6892
3	3	A B B A A B	8.68911
3	3	A B A B B A	8.68908
3	3	A B B B A A	8.6888
3	3	**A B A B A B**	8.6887
3	3	A A B B B A	8.6881
3	3	A B A A B B	8.6878
3	3	A A B B A B	8.6873
3	3	A A B A B B	8.6860
3	3	A A A B B B	8.6838
4	2	A A A A B B	8.6630
4	2	A A A B A B	8.6588
4	2	A A A B B A	8.6546
4	2	A A B A A B	8.6539
4	2	A A B A B A	8.6493
4	2	A B A A A B	8.6476
4	2	A A B B A A	8.6440
4	2	A B A A B A	8.6426
4	2	B A A A A B	8.6384
4	2	A B A B A A	8.6369
4	2	B A A A B A	8.6329
4	2	A B B A A A	8.6301
4	2	B A A B A A	8.6266
4	2	B A B A A A	8.6193
4	2	B B A A A A	8.6096
5	1	A A A A A B	8.5229
5	1	A A A A B A	8.5115
5	1	A A A B A A	8.4991
5	1	A A B A A A	8.4852
5	1	A B A A A A	8.4688
5	1	B A A A A A	8.4463
6	0	A A A A A A	8.1848

Table A.1.12 Ranking of pure selection scenarios for $n = 96$, $k = 6$, $v_a = v_b = 1$, and $T = 10^9$ runs.

The following table summarises the results of Tables A.1.7, A.1.10, A.1.11, and A.1.12. The scenarios are categorised using the A^iB^j-notation (i A-actions and j B-actions) with $i + j = k$ and $i \geq j$.

rank	$v_a = v_b = \frac{1}{8}$	$v_a = v_b = \frac{1}{4}$	$v_a = v_b = \frac{1}{2}$	$v_a = v_b = 1$
1	A^3B^3	A^3B^3	A^3B^3	A^3B^3
2	A^3B^3	A^3B^3	A^3B^3	A^3B^3
3	A^3B^3	A^3B^3	A^3B^3	A^3B^3
4	A^4B^2	///////////	A^3B^3	A^3B^3
5	///////////	A^3B^3	///////////	///////////
6	A^4B^2	A^3B^3	A^3B^3	A^3B^3
7	A^3B^3	A^4B^2	A^3B^3	A^3B^3
8	A^4B^2	A^4B^2	A^3B^3	A^3B^3
9	A^4B^2	A^3B^3	A^3B^3	A^3B^3
10	A^3B^3	A^3B^3	A^4B^2	A^3B^3
11	A^3B^3	A^4B^2	A^3B^3	A^4B^2
12	A^4B^2	A^4B^2	A^4B^2	A^4B^2
13	A^4B^2	A^3B^3	A^4B^2	A^4B^2
14	A^3B^3	A^4B^2	A^4B^2	A^4B^2
15	A^4B^2	A^4B^2	A^4B^2	A^4B^2
16	A^4B^2	A^3B^3	A^4B^2	A^4B^2
17	A^3B^3	A^4B^2	A^4B^2	A^4B^2
18	A^4B^2	A^4B^2	A^4B^2	A^4B^2
19	A^4B^2	A^4B^2	A^4B^2	A^4B^2
20	A^5B^1	A^4B^2	A^4B^2	A^4B^2
21	A^4B^2	A^4B^2	A^4B^2	A^4B^2
22	A^3B^3	A^4B^2	A^4B^2	A^4B^2
23	A^4B^2	A^4B^2	A^4B^2	A^4B^2
24	A^5B^1	A^5B^1	A^4B^2	A^4B^2
25	A^4B^2	A^4B^2	A^4B^2	A^4B^2
26	A^4B^2	A^5B^1	A^5B^1	A^5B^1
27	A^5B^1	A^4B^2	A^5B^1	A^5B^1
28	A^4B^2	A^5B^1	A^5B^1	A^5B^1
29	A^5B^1	A^5B^1	A^5B^1	A^5B^1
30	A^5B^1	A^5B^1	A^5B^1	A^5B^1
31	A^5B^1	A^5B^1	A^5B^1	A^5B^1
32	A^6B^0	A^6B^0	A^6B^0	A^6B^0

A^3B^3	A^4B^2	A^5B^1	A^6B^0

Table A.1.13 Ranking of A^iB^j-scenarios for $n = 96$, $k = 6$, $v_a = v_b \in \{\frac{1}{8}, \frac{1}{4}, \frac{1}{2}, 1\}$, and $T = 10^9$ runs. Hatched cells mean strictly alternating selection order ($ABABAB$).

A.2 Experts with Different Noise Levels

In Section 4.2 starting on page 42 we look at the ranking of pure selection scenarios for $n = 96$, $k = 4$, $v_a = \frac{1}{4}$, and $v_b = \frac{1}{2}$ only. This section contains additional results for $n = 96$, $k \in \{3, 4, 5, 6\}$, $v_a = \frac{1}{4}$, and $v_b \in \{\frac{1}{2}, \frac{3}{4}\}$. Scenarios AB and BA are highlighted in each table. The average values are also rounded to four decimals.

#A	#B	selection order			$\sum_{i \in S_3} x_i$
3	0	A	A	A	5.8616
2	1	B	A	A	5.8577
2	1	**A**	**B**	**A**	5.8338
2	1	A	A	B	5.8059
1	2	B	B	A	5.7167
1	2	**B**	**A**	**B**	5.6723
1	2	A	B	B	5.6085
0	3	B	B	B	5.3509

Table A.2.1 Ranking of pure selection scenarios for $n = 96$, $k = 3$, $v_a = \frac{1}{4}$, $v_b = \frac{1}{2}$, and $T = 10^9$ runs.

#A	#B	selection order			$\sum_{i \in S_3} x_i$
3	0	A	A	A	5.8616
2	1	B	A	A	5.7662
2	1	**A**	**B**	**A**	5.7202
2	1	A	A	B	5.6769
1	2	B	B	A	5.4921
1	2	**B**	**A**	**B**	5.4326
1	2	A	B	B	5.3507
0	3	B	B	B	4.9539

Table A.2.2 Ranking of pure selection scenarios for $n = 96$, $k = 3$, $v_a = \frac{1}{4}$, $v_b = \frac{3}{4}$, and $T = 10^9$ runs.

#A	#B	selection order				$\sum_{i \in S_4} x_i$
3	1	B	A	A	A	7.4785
3	1	A	B	A	A	7.4687
4	0	A	A	A	A	7.4564
3	1	A	A	B	A	7.4546
3	1	A	A	A	B	7.4370
2	2	B	B	A	A	7.4093
2	2	**B**	**A**	**B**	**A**	7.3843
2	2	B	A	A	B	7.3582
2	2	A	B	B	A	7.3483
2	2	**A**	**B**	**A**	**B**	7.3192
2	2	A	A	B	B	7.2815
1	3	B	B	B	A	7.2146
1	3	B	B	A	B	7.1727
1	3	B	A	B	B	7.1233
1	3	A	B	B	B	7.0559
0	4	B	B	B	B	6.8067

Table A.2.3 Ranking of pure selection scenarios for $n = 96$, $k = 4$, $v_a = \frac{1}{4}$, $v_b = \frac{1}{2}$, and $T = 10^9$ runs.

#A	#B	selection order				$\sum_{i \in S_4} x_i$
4	0	A	A	A	A	7.4564
3	1	B	A	A	A	7.4031
3	1	A	B	A	A	7.3723
3	1	A	A	B	A	7.3428
3	1	A	A	A	B	7.3123
2	2	B	B	A	A	7.2211
2	2	**B**	**A**	**B**	**A**	7.1807
2	2	B	A	A	B	7.1428
2	2	A	B	B	A	7.1260
2	2	**A**	**B**	**A**	**B**	7.0857
2	2	A	A	B	B	7.0354
1	3	B	B	B	A	6.8855
1	3	B	B	A	B	6.8314
1	3	B	A	B	B	6.7690
1	3	A	B	B	B	6.6852
0	4	B	B	B	B	6.3018

Table A.2.4 Ranking of pure selection scenarios for $n = 96$, $k = 4$, $v_a = \frac{1}{4}$, $v_b = \frac{3}{4}$, and $T = 10^9$ runs.

#A	#B	selection order	$\sum_{i \in S_5} x_i$
4	1	B A A A A	8.9837
4	1	A B A A A	8.9824
4	1	A A B A A	8.9764
4	1	A A A B A	8.9675
4	1	A A A A B	8.9556
3	2	B B A A A	8.9551
5	0	A A A A A	8.9476
3	2	B A B A A	8.9410
3	2	B A A B A	8.9254
3	2	A B B A A	8.9198
3	2	B A A A B	8.9079
3	2	**A B A B A**	8.9021
3	2	A B A A B	8.8827
3	2	A A B B A	8.8789
3	2	A A B A B	8.8580
2	3	B B B A A	8.8404
3	2	A A A B B	8.8318
2	3	B B A B A	8.8147
2	3	B B A A B	8.7890
2	3	B A B B A	8.7844
2	3	**B A B A B**	8.7568
2	3	A B B B A	8.7438
2	3	B A A B B	8.7255
2	3	A B B A B	8.7137
2	3	A B A B B	8.6801
2	3	A A B B B	8.6384
1	4	B B B B A	8.6073
1	4	B B B A B	8.5671
1	4	B B A B B	8.5230
1	4	B A B B B	8.4721
1	4	A B B B B	8.4046
0	5	B B B B B	8.1682

Table A.2.5 Ranking of pure selection scenarios for $n = 96$, $k = 5$, $v_a = \frac{1}{4}$, $v_b = \frac{1}{2}$, and $T = 10^9$ runs.

#A	#B	selection order	$\sum_{i \in S_5} x_i$
5	0	A A A A A	8.9480
4	1	B A A A A	8.9210
4	1	A B A A A	8.9001
4	1	A A B A A	8.8793
4	1	A A A B A	8.8575
4	1	A A A A B	8.8342
3	2	B B A A A	8.7949
3	2	B A B A A	8.7658
3	2	B A A B A	8.7383
3	2	A B B A A	8.7264
3	2	B A A A B	8.7103
3	2	**A B A B A**	8.6973
3	2	A B A A B	8.6679
3	2	A A B B A	8.6611
3	2	A A B A B	8.6306
3	2	A A A B B	8.5937
2	3	B B B A A	8.5582
2	3	B B A B A	8.5199
2	3	B B A A B	8.4842
2	3	B A B B A	8.4762
2	3	**B A B A B**	8.4390
2	3	A B B B A	8.4187
2	3	B A A B B	8.3977
2	3	A B B A B	8.3795
2	3	A B A B B	8.3363
2	3	A A B B B	8.2833
1	4	B B B B A	8.1799
1	4	B B B A B	8.1291
1	4	B B A B B	8.0740
1	4	B A B B B	8.0112
1	4	A B B B B	7.9281
0	5	B B B B B	7.5624

Table A.2.6 Ranking of pure selection scenarios for $n = 96$, $k = 5$, $v_a = \frac{1}{4}$, $v_b = \frac{3}{4}$, and $T = 10^9$ runs.

#A	#B	selection order	$\sum_{i \in S_6} x_i$
5	1	A B A A A A	10.4004
5	1	A A B A A A	10.3997
5	1	B A A A A A	10.3964
5	1	A A A B A A	10.3963
4	2	B B A A A A	10.3934
5	1	A A A A B A	10.3905
4	2	B A B A A A	10.3861
5	1	A A A A A B	10.3824
4	2	B A A B A A	10.3772
4	2	A B B A A A	10.3740
4	2	B A A A B A	10.3666
4	2	A B A B A A	10.3634
4	2	B A A A A B	10.3542
6	0	A A A A A A	10.3526
4	2	A B A A B A	10.3513
4	2	A A B B A A	10.3491
4	2	A B A A A B	10.3376
4	2	A A B A B A	10.3358
3	3	B B B A A A	10.3261
4	2	A A B A A B	10.3209
4	2	A A A B B A	10.3191
3	3	B B A B A A	10.3097
4	2	A A A B A B	10.3031
3	3	B B A A B A	10.2929
3	3	B A B B A A	10.2901
4	2	A A A A B B	10.2835
3	3	B B A A A B	10.2750
3	3	**B A B A B A**	10.2721
3	3	A B B B A A	10.2638
3	3	B A B A A B	10.2530
3	3	B A A B B A	10.2516
3	3	A B B A B A	10.2441
3	3	B A A B A B	10.2315
3	3	A B B A A B	10.2236
3	3	A B A B B A	10.2220
3	3	B A A A B B	10.2088
3	3	**A B A B A B**	10.2004
3	3	A A B B B A	10.1948
2	4	B B B B A A	10.1771
3	3	A B A A B B	10.1764
3	3	A A B B A B	10.1719
2	4	B B B A B A	10.1510
3	3	A A B A B B	10.1467
2	4	B B B A A B	10.1256
2	4	B B A B B A	10.1225
3	3	A A A B B B	10.1165
2	4	B B A B A B	10.0958
2	4	B A B B B A	10.0901
2	4	B B A A B B	10.0670
2	4	B A B B A B	10.0618
2	4	A B B B B A	10.0481
2	4	B A B A B B	10.0315
2	4	A B B B A B	10.0176
2	4	B A A B B B	9.9978
2	4	A B B A B B	9.9853
2	4	A B A B B B	9.9497
1	5	B B B B B A	9.9147
2	4	A A B B B B	9.9064
1	5	B B B B A B	9.8758
1	5	B B B A B B	9.8345
1	5	B B A B B B	9.7899
1	5	B A B B B B	9.7392
1	5	A B B B B B	9.6731
0	6	B B B B B B	9.4510

Table A.2.7 Ranking of pure selection scenarios for $n = 96$, $k = 6$, $v_a = \frac{1}{4}$, $v_b = \frac{1}{2}$, and $T = 10^8$ runs.

#A	#B	selection order	$\sum_{i \in S_6} x_i$	#A	#B	selection order	$\sum_{i \in S_6} x_i$
6	0	A A A A A A	10.3530	3	3	B A A B A B	9.9317
5	1	B A A A A A	10.3437	3	3	A B B A A B	9.9181
5	1	A B A A A A	10.3295	3	3	A B A B B A	9.9151
5	1	A A B A A A	10.3148	3	3	B A A A B B	9.9002
5	1	A A A B A A	10.2990	3	3	**A B A B A B**	9.8852
5	1	A A A A B A	10.2819	3	3	A A B B B A	9.8761
5	1	A A A A A B	10.2633	3	3	A B A A B B	9.8526
4	2	B B A A A A	10.2552	3	3	A A B B A B	9.8452
4	2	B A B A A A	10.2337	3	3	A A B A B B	9.8117
4	2	B A A B A A	10.2131	2	4	B B B B A A	9.8046
4	2	A B B A A A	10.2044	3	3	A A A B B B	9.7721
4	2	B A A A B A	10.1921	2	4	B B B A B A	9.7676
4	2	A B A B A A	10.1826	2	4	B B B A A B	9.7332
4	2	B A A A A B	10.1700	2	4	B B A B B A	9.7279
4	2	A B A A B A	10.1605	2	4	B B A B A B	9.6924
4	2	A A B B A A	10.1555	2	4	B A B B B A	9.6833
4	2	A B A A A B	10.1375	2	4	B B A A B B	9.6549
4	2	A A B A B A	10.1325	2	4	B A B B A B	9.6465
4	2	A A B A A B	10.1087	2	4	A B B B B A	9.6254
4	2	A A A B B A	10.1048	2	4	B A B A B B	9.6077
3	3	B B B A A A	10.0805	2	4	A B B B A B	9.5868
4	2	A A A B A B	10.0802	2	4	B A A B B B	9.5649
3	3	B B A B A A	10.0518	2	4	A B B A B B	9.5462
4	2	A A A A B B	10.0512	2	4	A B A B B B	9.5020
3	3	B B A A B A	10.0250	2	4	A A B B B B	9.4482
3	3	B A B B A A	10.0191	1	5	B B B B B A	9.3939
3	3	B B A A A B	9.9979	1	5	B B B B A B	9.3453
3	3	**B A B A B A**	9.9913	1	5	B B B A B B	9.2944
3	3	A B B B A A	9.9764	1	5	B B A B B B	9.2396
3	3	B A B A A B	9.9634	1	5	B A B B B B	9.1776
3	3	B A A B B A	9.9605	1	5	A B B B B B	9.0965
3	3	A B B A B A	9.9471	0	6	B B B B B B	8.7499

Table A.2.8 Ranking of pure selection scenarios for $n = 96$, $k = 6$, $v_a = \frac{1}{4}$, $v_b = \frac{3}{4}$, and $T = 10^9$ runs.

B The Selection Problem with k Experts

This chapter contains rankings of k-SeP for $k \in \{3, 4, 5, 6\}$, $n \in \{k+1, 2k, 3k, 4k\}$, and the first and the second distribution for the noise levels. For $k \in \{5, 6\}$ the rankings are shortened to the ten best and worst selection orders.

selection order			$\sum_{i \in S_3} x_i$
3	2	1	0.8920
2	3	1	0.8837
3	1	2	0.8593
1	3	2	0.8388
2	1	3	0.8335
1	2	3	0.8206

$n = 4$

selection order			$\sum_{i \in S_3} x_i$
3	2	1	1.8165
2	3	1	1.8005
3	1	2	1.7693
2	1	3	1.7302
1	3	2	1.7280
1	2	3	1.7036

$n = 6$

selection order			$\sum_{i \in S_3} x_i$
3	2	1	2.5576
2	3	1	2.5375
3	1	2	2.5063
2	1	3	2.4633
1	3	2	2.4533
1	2	3	2.4284

$n = 9$

selection order			$\sum_{i \in S_3} x_i$
3	2	1	3.0156
2	3	1	2.9942
3	1	2	2.9639
2	1	3	2.9206
1	3	2	2.9062
1	2	3	2.8821

$n = 12$

Table B.1 Ranking of all selection orders for $k = 3$, $n \in \{4, 6, 9, 12\}$, $v = \left(\frac{1}{3}, \frac{2}{3}, \frac{3}{3}\right) = \left(\frac{1}{3}, \frac{2}{3}, 1\right)$, and $T = 10^9$ runs.

selection order			$\sum_{i \in S_3} x_i$
3	2	1	0.9165
2	3	1	0.9029
3	1	2	0.8853
1	3	2	0.8616
2	1	3	0.8423
1	2	3	0.8308

$n = 4$

selection order			$\sum_{i \in S_3} x_i$
3	2	1	1.8658
2	3	1	1.8392
3	1	2	1.8195
1	3	2	1.7710
2	1	3	1.7539
1	2	3	1.7294

$n = 6$

selection order			$\sum_{i \in S_3} x_i$
3	2	1	2.6265
2	3	1	2.5926
3	1	2	2.5752
1	3	2	2.5119
2	1	3	2.5026
1	2	3	2.4695

$n = 9$

selection order			$\sum_{i \in S_3} x_i$
3	2	1	3.0966
2	3	1	3.0601
3	1	2	3.0442
1	3	2	2.9748
2	1	3	2.9706
1	2	3	2.9336

$n = 12$

Table B.2 Ranking of all selection orders for $k = 3$, $n \in \{4, 6, 9, 12\}$, $v = \left(\frac{1}{4}, \frac{2}{4}, \frac{4}{4}\right) = \left(\frac{1}{4}, \frac{1}{2}, 1\right)$, and $T = 10^9$ runs.

selection order				$\sum_{i \in S_4} x_i$
4	3	2	1	1.0402
3	4	2	1	1.0374
4	2	3	1	1.0329
2	4	3	1	1.0270
3	2	4	1	1.0248
2	3	4	1	1.0216
4	3	1	2	1.0107
3	4	1	2	1.0078
4	1	3	2	0.9930
4	2	1	3	0.9869
3	1	4	2	0.9849
1	4	3	2	0.9834
2	4	1	3	0.9803
1	3	4	2	0.9781
4	1	2	3	0.9759
3	2	1	4	0.9680
1	4	2	3	0.9657
2	3	1	4	0.9643
2	1	4	3	0.9568
3	1	2	4	0.9566
1	2	4	3	0.9530
1	3	2	4	0.9488
2	1	3	4	0.9453
1	2	3	4	0.9413

$n = 5$

selection order				$\sum_{i \in S_4} x_i$
4	3	2	1	2.5719
3	4	2	1	2.5640
4	2	3	1	2.5558
2	4	3	1	2.5380
3	2	4	1	2.5371
2	3	4	1	2.5271
4	3	1	2	2.5269
3	4	1	2	2.5189
4	2	1	3	2.4887
4	1	3	2	2.4860
2	4	1	3	2.4698
3	1	4	2	2.4680
4	1	2	3	2.4625
3	2	1	4	2.4568
1	4	3	2	2.4551
2	3	1	4	2.4458
1	3	4	2	2.4445
1	4	2	3	2.4305
3	1	2	4	2.4300
2	1	4	3	2.4188
1	3	2	4	2.4050
1	2	4	3	2.4046
2	1	3	4	2.4039
1	2	3	4	2.3893

$n = 8$

selection order				$\sum_{i \in S_4} x_i$
4	3	2	1	3.5886
3	4	2	1	3.5782
4	2	3	1	3.5693
3	2	4	1	3.5467
2	4	3	1	3.5450
4	3	1	2	3.5414
2	3	4	1	3.5325
3	4	1	2	3.5307
4	2	1	3	3.5007
4	1	3	2	3.4910
2	4	1	3	3.4752
3	1	4	2	3.4697
4	1	2	3	3.4678
3	2	1	4	3.4660
2	3	1	4	3.4509
1	4	3	2	3.4469
1	3	4	2	3.4351
3	1	2	4	3.4327
1	4	2	3	3.4227
2	1	4	3	3.4147
2	1	3	4	3.4005
1	3	2	4	3.3967
1	2	4	3	3.3932
1	2	3	4	3.3786

$n = 12$

selection order				$\sum_{i \in S_4} x_i$
4	3	2	1	4.2138
3	4	2	1	4.2024
4	2	3	1	4.1936
3	2	4	1	4.1697
4	3	1	2	4.1667
2	4	3	1	4.1663
3	4	1	2	4.1550
2	3	4	1	4.1536
4	2	1	3	4.1261
4	1	3	2	4.1128
2	4	1	3	4.0978
3	2	1	4	4.0913
3	1	4	2	4.0906
4	1	2	3	4.0904
2	3	1	4	4.0743
1	4	3	2	4.0624
3	1	2	4	4.0554
1	3	4	2	4.0504
1	4	2	3	4.0391
2	1	4	3	4.0344
2	1	3	4	4.0210
1	3	2	4	4.0139
1	2	4	3	4.0090
1	2	3	4	3.9952

$n = 16$

Table B.3 Ranking of all selection orders for $k = 4$, $n \in \{5, 8, 12, 16\}$, $v = \left(\frac{1}{4}, \frac{2}{4}, \frac{3}{4}, \frac{4}{4}\right) = \left(\frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1\right)$, and $T = 10^9$ runs.

selection order				$\sum_{i \in S_4} x_i$
4	3	2	1	1.0735
3	4	2	1	1.0674
4	2	3	1	1.0673
2	4	3	1	1.0590
3	2	4	1	1.0489
4	3	1	2	1.0471
2	3	4	1	1.0466
3	4	1	2	1.0408
4	1	3	2	1.0331
4	2	1	3	1.0246
1	4	3	2	1.0226
4	1	2	3	1.0164
2	4	1	3	1.0155
3	1	4	2	1.0147
1	3	4	2	1.0105
1	4	2	3	1.0052
2	1	4	3	0.9890
1	2	4	3	0.9868
3	2	1	4	0.9808
2	3	1	4	0.9779
3	1	2	4	0.9714
1	3	2	4	0.9657
2	1	3	4	0.9616
1	2	3	4	0.9589

$n = 5$

selection order				$\sum_{i \in S_4} x_i$
4	3	2	1	2.6570
4	2	3	1	2.6421
3	4	2	1	2.6392
2	4	3	1	2.6165
4	3	1	2	2.6151
3	2	4	1	2.5991
3	4	1	2	2.5967
2	3	4	1	2.5909
4	1	3	2	2.5797
4	2	1	3	2.5773
4	1	2	3	2.5557
2	4	1	3	2.5501
1	4	3	2	2.5450
3	1	4	2	2.5378
1	3	4	2	2.5203
1	4	2	3	2.5199
3	2	1	4	2.5032
2	3	1	4	2.4937
2	1	4	3	2.4912
1	2	4	3	2.4814
3	1	2	4	2.4797
1	3	2	4	2.4597
2	1	3	4	2.4556
1	2	3	4	2.4447

$n = 8$

selection order				$\sum_{i \in S_4} x_i$
4	3	2	1	3.7098
4	2	3	1	3.6912
3	4	2	1	3.6855
4	3	1	2	3.6647
2	4	3	1	3.6552
3	4	1	2	3.6398
3	2	4	1	3.6381
2	3	4	1	3.6258
4	2	1	3	3.6233
4	1	3	2	3.6190
4	1	2	3	3.5948
2	4	1	3	3.5856
1	4	3	2	3.5688
3	1	4	2	3.5685
1	4	2	3	3.5433
3	2	1	4	3.5421
1	3	4	2	3.5407
2	3	1	4	3.5284
2	1	4	3	3.5151
3	1	2	4	3.5120
1	2	4	3	3.4988
1	3	2	4	3.4816
2	1	3	4	3.4811
1	2	3	4	3.4637

$n = 12$

selection order				$\sum_{i \in S_4} x_i$
4	3	2	1	4.3579
4	2	3	1	4.3377
3	4	2	1	4.3307
4	3	1	2	4.3121
2	4	3	1	4.2970
3	4	1	2	4.2844
3	2	4	1	4.2813
4	2	1	3	4.2700
2	3	4	1	4.2667
4	1	3	2	4.2619
4	1	2	3	4.2383
2	4	1	3	4.2276
3	1	4	2	4.2089
1	4	3	2	4.2039
3	2	1	4	4.1880
1	4	2	3	4.1790
1	3	4	2	4.1752
2	3	1	4	4.1722
3	1	2	4	4.1550
2	1	4	3	4.1533
1	2	4	3	4.1332
2	1	3	4	4.1212
1	3	2	4	4.1189
1	2	3	4	4.1001

$n = 16$

Table B.4 Ranking of all selection orders for $k = 4$, $n \in \{5, 8, 12, 16\}$, $v = \left(\frac{1}{6}, \frac{2}{6}, \frac{3}{6}, \frac{6}{6}\right) = \left(\frac{1}{6}, \frac{1}{3}, \frac{1}{2}, 1\right)$, and $T = 10^9$ runs.

selection order					$\sum_{i \in S_5} x_i$
5	4	3	2	1	1.1565
4	5	3	2	1	1.1554
5	3	4	2	1	1.1541
3	5	4	2	1	1.1517
4	3	5	2	1	1.1508
5	4	2	3	1	1.1499
3	4	5	2	1	1.1497
4	5	2	3	1	1.1488
5	2	4	3	1	1.1448
5	3	2	4	1	1.1425
⋮					
3	2	1	4	5	1.04533
2	1	3	5	4	1.04531
1	2	3	5	4	1.0444
2	3	1	4	5	1.0438
2	1	4	3	5	1.0420
3	1	2	4	5	1.0416
1	2	4	3	5	1.0411
1	3	2	4	5	1.0391
2	1	3	4	5	1.0369
1	2	3	4	5	1.0359

$n = 6$

selection order					$\sum_{i \in S_5} x_i$
5	4	3	2	1	3.3368
4	5	3	2	1	3.3323
5	3	4	2	1	3.3290
5	4	2	3	1	3.3211
3	5	4	2	1	3.3192
4	3	5	2	1	3.3184
4	5	2	3	1	3.3165
3	4	5	2	1	3.3131
5	2	4	3	1	3.3033
5	3	2	4	1	3.3026
⋮					
1	4	2	3	5	3.1127
3	1	2	4	5	3.1094
1	3	2	5	4	3.1052
2	1	3	5	4	3.1023
2	1	4	3	5	3.1022
1	2	3	5	4	3.0950
1	2	4	3	5	3.0949
1	3	2	4	5	3.0947
2	1	3	4	5	3.0918
1	2	3	4	5	3.0844

$n = 10$

selection order					$\sum_{i \in S_5} x_i$
5	4	3	2	1	4.6305
4	5	3	2	1	4.6241
5	3	4	2	1	4.6204
5	4	2	3	1	4.6122
4	3	5	2	1	4.6067
3	5	4	2	1	4.6063
4	5	2	3	1	4.6058
3	4	5	2	1	4.5989
5	3	2	4	1	4.5905
5	2	4	3	1	4.5888
⋮					
1	4	2	3	5	4.3722
1	2	5	3	4	4.3676
2	1	4	3	5	4.3627
1	3	2	5	4	4.3614
2	1	3	5	4	4.3612
1	3	2	4	5	4.3518
2	1	3	4	5	4.3517
1	2	4	3	5	4.3499
1	2	3	5	4	4.3483
1	2	3	4	5	4.3386

$n = 15$

selection order					$\sum_{i \in S_5} x_i$
5	4	3	2	1	5.4239
4	5	3	2	1	5.4168
5	3	4	2	1	5.4131
5	4	2	3	1	5.4049
4	3	5	2	1	5.3982
4	5	2	3	1	5.3977
3	5	4	2	1	5.3970
3	4	5	2	1	5.3891
5	3	2	4	1	5.3824
5	4	3	1	2	5.3807
⋮					
1	2	4	5	3	5.1518
1	2	5	3	4	5.1463
2	1	4	3	5	5.1458
2	1	3	5	4	5.1437
1	3	2	5	4	5.1417
2	1	3	4	5	5.1349
1	3	2	4	5	5.1327
1	2	4	3	5	5.1296
1	2	3	5	4	5.1274
1	2	3	4	5	5.1184

$n = 20$

Table B.5 Ranking of all selection orders for $k = 5$, $n \in \{6, 10, 15, 20\}$, $v = \left(\frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, \frac{5}{5}\right) = \left(\frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, 1\right)$, and $T = 10^9$ runs.

selection order					$\sum_{i \in S_5} x_i$
5	4	3	2	1	1.1926
5	3	4	2	1	1.1908
4	5	3	2	1	1.1895
5	4	2	3	1	1.1873
3	5	4	2	1	1.1870
4	5	2	3	1	1.1842
5	2	4	3	1	1.1837
4	3	5	2	1	1.1817
5	3	2	4	1	1.18112
3	4	5	2	1	1.18106
⋮					
4	1	2	3	5	1.0707
1	4	2	3	5	1.0684
3	2	1	4	5	1.0652
2	1	4	3	5	1.06434
2	3	1	4	5	1.06430
1	2	4	3	5	1.0640
3	1	2	4	5	1.0629
1	3	2	4	5	1.0616
2	1	3	4	5	1.0597
1	2	3	4	5	1.0593

$n = 6$

selection order					$\sum_{i \in S_5} x_i$
5	4	3	2	1	3.4496
5	3	4	2	1	3.4428
4	5	3	2	1	3.4365
5	4	2	3	1	3.4356
3	5	4	2	1	3.4258
4	5	2	3	1	3.4224
5	2	4	3	1	3.4212
5	3	2	4	1	3.4185
4	3	5	2	1	3.4118
5	4	3	1	2	3.4115
⋮					
1	2	3	5	4	3.2036
3	2	1	4	5	3.1999
1	4	2	3	5	3.1965
2	3	1	4	5	3.1946
3	1	2	4	5	3.1888
2	1	4	3	5	3.1851
1	2	4	3	5	3.1812
1	3	2	4	5	3.1793
2	1	3	4	5	3.1749
1	2	3	4	5	3.1708

$n = 10$

selection order					$\sum_{i \in S_5} x_i$
5	4	3	2	1	4.7935
5	3	4	2	1	4.7842
5	4	2	3	1	4.7766
4	5	3	2	1	4.7746
3	5	4	2	1	4.7592
4	5	2	3	1	4.7576
5	2	4	3	1	4.7562
5	3	2	4	1	4.7555
5	4	3	1	2	4.7534
5	2	3	4	1	4.7444
⋮					
2	1	3	5	4	4.5062
2	3	1	4	5	4.5050
1	4	2	3	5	4.5005
1	2	3	5	4	4.4987
3	1	2	4	5	4.4970
2	1	4	3	5	4.4880
1	3	2	4	5	4.4803
1	2	4	3	5	4.4802
2	1	3	4	5	4.4768
1	2	3	4	5	4.4686

$n = 15$

selection order					$\sum_{i \in S_5} x_i$
5	4	3	2	1	5.6193
5	3	4	2	1	5.6088
5	4	2	3	1	5.6013
4	5	3	2	1	5.5976
3	5	4	2	1	5.5797
4	5	2	3	1	5.5794
5	4	3	1	2	5.5789
5	3	2	4	1	5.5787
5	2	4	3	1	5.5779
5	3	4	1	2	5.5682
⋮					
1	3	2	5	4	5.3175
2	1	3	5	4	5.3151
3	1	2	4	5	5.3113
1	4	2	3	5	5.3111
1	2	3	5	4	5.3048
2	1	4	3	5	5.2991
1	3	2	4	5	5.2900
1	2	4	3	5	5.2886
2	1	3	4	5	5.2878
1	2	3	4	5	5.2769

$n = 20$

Table B.6 Ranking of all selection orders for $k = 5$, $n \in \{6, 10, 15, 20\}$, $v = \left(\frac{1}{8}, \frac{2}{8}, \frac{3}{8}, \frac{4}{8}, \frac{8}{8}\right) = \left(\frac{1}{8}, \frac{1}{4}, \frac{3}{8}, \frac{1}{2}, 1\right)$, and $T = 10^9$ runs.

selection order						$\sum_{i \in S_6} x_i$
6	5	4	3	2	1	1.2513
5	6	4	3	2	1	1.2508
6	4	5	3	2	1	1.2503
4	6	5	3	2	1	1.2492
6	5	3	4	2	1	1.2491
5	4	6	3	2	1	1.2488
5	6	3	4	2	1	1.2486
4	5	6	3	2	1	1.2483
6	3	5	4	2	1	1.2470
6	4	3	5	2	1	1.2462
⋮						
2	1	3	5	4	6	1.1173
1	2	3	5	4	6	1.1172
3	2	1	4	5	6	1.1161
2	1	4	3	5	6	1.1158
1	2	4	3	5	6	1.1157
2	3	1	4	5	6	1.1156
3	1	2	4	5	6	1.1152
1	3	2	4	5	6	1.1146
2	1	3	4	5	6	1.11341
1	2	3	4	5	6	1.11337

$n = 7$

selection order						$\sum_{i \in S_6} x_i$
6	5	4	3	2	1	4.1073
5	6	4	3	2	1	4.1044
6	4	5	3	2	1	4.1028
6	5	3	4	2	1	4.0996
5	6	3	4	2	1	4.096603
4	6	5	3	2	1	4.096596
5	4	6	3	2	1	4.0960
4	5	6	3	2	1	4.0928
6	5	4	2	3	1	4.0920
6	3	5	4	2	1	4.0897
⋮						
1	3	2	5	4	6	3.8001
2	1	3	4	6	5	3.7970
2	1	3	5	4	6	3.7968
2	1	4	3	5	6	3.7966
1	2	3	4	6	5	3.7937
1	2	3	5	4	6	3.7935
1	2	4	3	5	6	3.7933
1	3	2	4	5	6	3.7925
2	1	3	4	5	6	3.7892
1	2	3	4	5	6	3.7858

$n = 12$

selection order						$\sum_{i \in S_6} x_i$
6	5	4	3	2	1	5.6789
5	6	4	3	2	1	5.6746
6	4	5	3	2	1	5.6727
6	5	3	4	2	1	5.6693
5	6	3	4	2	1	5.6650
5	4	6	3	2	1	5.66359
4	6	5	3	2	1	5.66355
6	5	4	2	3	1	5.6616
4	5	6	3	2	1	5.6586
5	6	4	2	3	1	5.6573
⋮						
2	1	4	3	5	6	5.3214
1	2	4	3	6	5	5.3213
2	1	3	5	4	6	5.3205
2	1	3	4	6	5	5.3197
1	3	2	4	5	6	5.3150
1	2	4	3	5	6	5.3143
1	2	3	5	4	6	5.3133
2	1	3	4	5	6	5.3128
1	2	3	4	6	5	5.3125
1	2	3	4	5	6	5.3055

$n = 18$

selection order						$\sum_{i \in S_6} x_i$
6	5	4	3	2	1	6.6414
5	6	4	3	2	1	6.6365
6	4	5	3	2	1	6.6346
6	5	3	4	2	1	6.6311
5	6	3	4	2	1	6.6262
5	4	6	3	2	1	6.6244
4	6	5	3	2	1	6.6239
6	5	4	2	3	1	6.6236
5	6	4	2	3	1	6.6187
4	5	6	3	2	1	6.6186
⋮						
2	1	3	5	4	6	6.2662
1	3	2	4	6	5	6.2660
2	1	3	4	6	5	6.2651
1	2	4	3	6	5	6.2643
1	3	2	4	5	6	6.2597
2	1	3	4	5	6	6.2588
1	2	4	3	5	6	6.2579
1	2	3	5	4	6	6.2563
1	2	3	4	6	5	6.2551
1	2	3	4	5	6	6.2487

$n = 24$

Table B.7 Ranking of all selection orders for $k = 6$, $n \in \{7, 12, 18, 24\}$, $v = \left(\frac{1}{6}, \frac{2}{6}, \frac{3}{6}, \frac{4}{6}, \frac{5}{6}, \frac{6}{6}\right) = \left(\frac{1}{6}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{5}{6}, 1\right)$, and $T = 10^9$ runs.

selection order						$\sum_{i\in S_6} x_i$
6	5	4	3	2	1	1.2878
6	4	5	3	2	1	1.2871
6	5	3	4	2	1	1.2862
5	6	4	3	2	1	1.2860
4	6	5	3	2	1	1.2851
6	3	5	4	2	1	1.2849
5	6	3	4	2	1	1.2844
6	4	3	5	2	1	1.2840
6	3	4	5	2	1	1.2834
6	5	4	2	3	1	1.2831
⋮						
2	1	3	5	4	6	1.1447
1	4	2	3	5	6	1.1445
1	2	4	3	5	6	1.1430
2	1	4	3	5	6	1.1429
3	2	1	4	5	6	1.1423
2	3	1	4	5	6	1.1421
3	1	2	4	5	6	1.1420
1	3	2	4	5	6	1.1418
1	2	3	4	5	6	1.1412
2	1	3	4	5	6	1.1411

$n = 7$

selection order						$\sum_{i\in S_6} x_i$
6	5	4	3	2	1	4.2426
6	4	5	3	2	1	4.2389
6	5	3	4	2	1	4.2361
5	6	4	3	2	1	4.2324
6	5	4	2	3	1	4.2295
6	3	5	4	2	1	4.2286
6	4	3	5	2	1	4.2269
4	6	5	3	2	1	4.2265
5	6	3	4	2	1	4.2259
6	4	5	2	3	1	4.2257
⋮						
1	3	2	5	4	6	3.9160
2	3	1	4	5	6	3.9143
2	1	3	5	4	6	3.9125
3	1	2	4	5	6	3.9123
1	2	3	5	4	6	3.9115
2	1	4	3	5	6	3.9113
1	2	4	3	5	6	3.9103
1	3	2	4	5	6	3.9083
2	1	3	4	5	6	3.9048
1	2	3	4	5	6	3.9037

$n = 12$

selection order						$\sum_{i\in S_6} x_i$
6	5	4	3	2	1	5.8770
6	4	5	3	2	1	5.8716
6	5	3	4	2	1	5.8684
5	6	4	3	2	1	5.8616
6	5	4	2	3	1	5.8615
6	3	5	4	2	1	5.8571
6	4	3	5	2	1	5.8562
6	4	5	2	3	1	5.8560
5	6	3	4	2	1	5.8530
4	6	5	3	2	1	5.8525
⋮						
2	3	1	4	5	6	5.4996
3	1	2	4	5	6	5.4962
1	3	2	5	4	6	5.4957
2	1	4	3	5	6	5.4917
2	1	3	5	4	6	5.4916
1	2	4	3	5	6	5.4886
1	2	3	5	4	6	5.4885
1	3	2	4	5	6	5.4876
2	1	3	4	5	6	5.4836
1	2	3	4	5	6	5.4804

$n = 18$

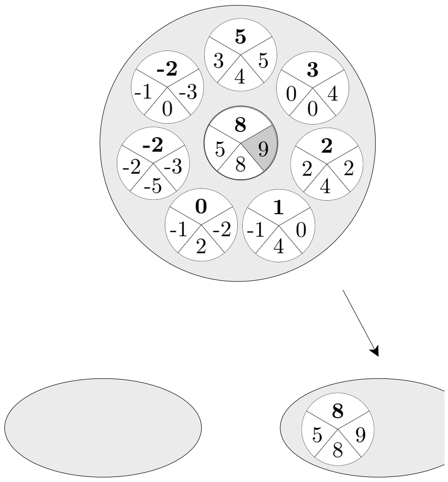
selection order						$\sum_{i\in S_6} x_i$
6	5	4	3	2	1	6.8805
6	4	5	3	2	1	6.8742
6	5	3	4	2	1	6.8710
6	5	4	2	3	1	6.8642
5	6	4	3	2	1	6.8625
6	4	5	2	3	1	6.8578
6	3	5	4	2	1	6.8576
6	4	3	5	2	1	6.8573
5	6	3	4	2	1	6.8529
4	6	5	3	2	1	6.8516
⋮						
1	2	5	3	4	6	6.4841
3	1	2	4	5	6	6.4811
1	3	2	5	4	6	6.4772
2	1	4	3	5	6	6.4741
2	1	3	5	4	6	6.4734
1	2	4	3	5	6	6.4693
1	3	2	4	5	6	6.4692
1	2	3	5	4	6	6.4686
2	1	3	4	5	6	6.4655
1	2	3	4	5	6	6.4605

$n = 24$

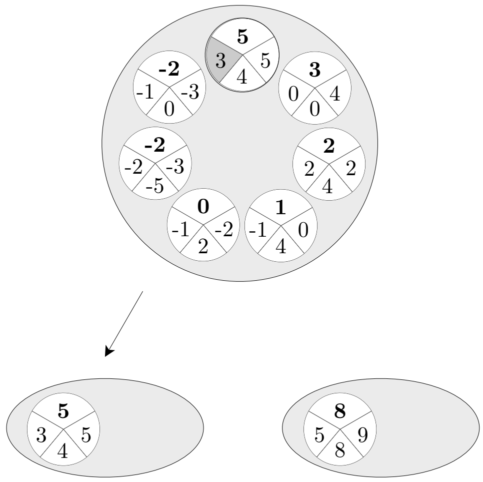
Table B.8 Ranking of all selection orders for $k = 6$, $n \in \{7, 12, 18, 24\}$, $v = \left(\frac{1}{10}, \frac{2}{10}, \frac{3}{10}, \frac{4}{10}, \frac{5}{10}, \frac{10}{10}\right) = \left(\frac{1}{10}, \frac{1}{5}, \frac{3}{10}, \frac{2}{5}, \frac{1}{2}, 1\right)$, and $T = 10^9$ runs.

C The Team Selection Problem

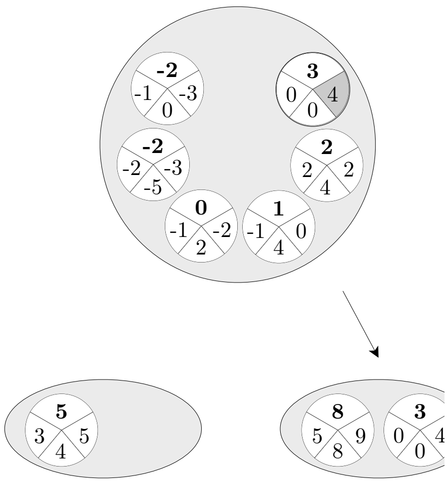
The following figures illustrate scenario $C:AB$ of the Team Selection Problem with $n = 8$ and $k = 2$. It extends the example on page 21. Here, the single-expert-team C wins by $11:7$.



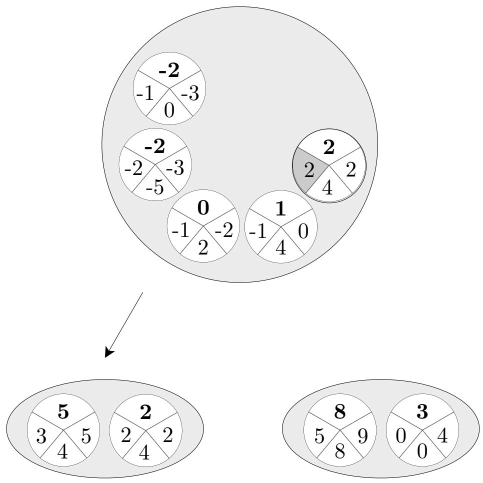
Step 1 (action "C"): Expert C selects item 1.



Step 2 (action "A"): Expert A selects item 2.



Step 3 (action "C"): Expert C selects item 3.



Step 4 (action "B"): Expert B selects item 4.

As already explained in Chapter 6 starting on page 51, for equally good experts A, B, and C team AB is better than team C with scenario $AB : C$ for all considered noise levels $v_a = v_b = v_c \in \{\frac{1}{8}, \frac{1}{4}, \frac{1}{2}, 1\}$. Considering a fair competition of this scenario, team AB is also better than team C for these noise levels. Both observations are presented in the following figures.

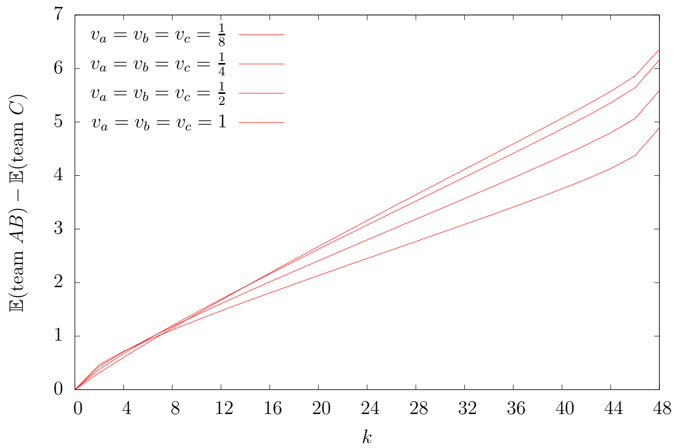


Figure C.1 Scenario $AB : C$ for $n = 96$, $v_a = v_b = v_c \in \{\frac{1}{8}, \frac{1}{4}, \frac{1}{2}, 1\}$, and $T = 10^8$ runs.

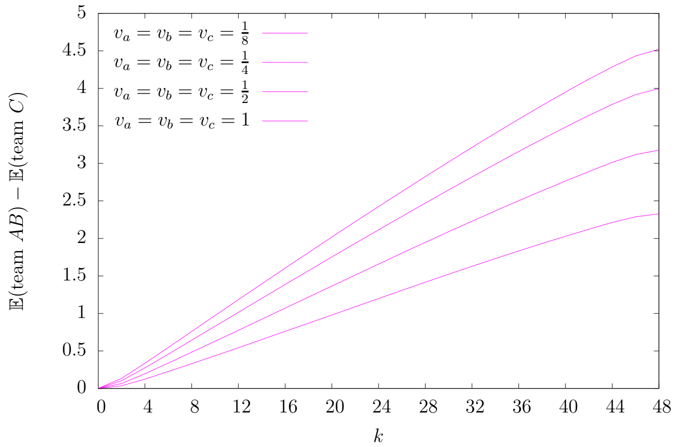


Figure C.2 A fair competition between team AB and team C ($\frac{T}{2}$ times order $ACBC$ and $\frac{T}{2}$ times order $CACB$) for $n = 96$, $v_a = v_b = v_c \in \{\frac{1}{8}, \frac{1}{4}, \frac{1}{2}, 1\}$, and $T = 10^8$ runs.

D The Group Selection Problem

D.1 The Group Selection Problem with Four Experts

In addition to Model 2.4.3 introduced in Section 2.4 on page 24 these are all 36 scenarios considered in 4-GSeP.

$3 \cdot 2! \cdot 2! = 12$ scenarios of $(2;2)$-type:

$AB;CD$	$AB;DC$	$AC;BD$	$AC;DB$	$AD;BC$	$AD;CB$
$BA;CD$	$BA;DC$	$CA;BD$	$CA;DB$	$DA;BC$	$DA;CB$

$4 \cdot 1! \cdot 3! = 24$ scenarios of $(1;3)$-type:

$A;BCD$	$A;BDC$	$A;CBD$	$A;CDB$	$A;DBC$	$A;DCB$
$B;ACD$	$B;ADC$	$B;CAD$	$B;CDA$	$B;DAC$	$B;DCA$
$C;ABD$	$C;ADB$	$C;BAD$	$C;BDA$	$C;DAB$	$C;DBA$
$D;ABC$	$D;ACB$	$D;BAC$	$D;BCA$	$D;CAB$	$D;CBA$

The following three tables include rankings of scenarios of $(1;3)$-type and rankings of scenarios of $(1;3)$- and $(2;2)$-type for several n, k, and noise levels.

scenario	$v^{(1)} = \left(\frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1\right)$		$v^{(2)} = \left(\frac{1}{4}, \frac{5}{16}, \frac{15}{16}, 1\right)$		$v^{(3)} = \left(\frac{1}{4}, \frac{9}{16}, \frac{5}{8}, 1\right)$		$v^{(4)} = \left(\frac{1}{4}, \frac{5}{16}, \frac{3}{8}, 1\right)$		$v^{(5)} = \left(\frac{1}{4}, \frac{7}{8}, \frac{15}{16}, 1\right)$	
	rank	total sum	rank	total sum	rank	total sum	rank	total sum	rank	total sum
$A; DCB$	1	1.7812	1	1.8050	1	1.7781	1	1.8306	1	1.7280
$A; CDB$	2	1.7752	2	1.8036	3	1.7684	4	1.8125	2	1.7266
$A; DBC$	3	1.7616	5	1.7539	2	1.7729	2	1.8224	3	1.7248
$A; CBD$	6	1.7426	6	1.7499	5	1.7416	13	1.7611	5	1.7203
$A; BDC$	4	1.7475	9	1.7316	4	1.7612	8	1.8017	4	1.7218
$A; BCD$	8	1.7343	10	1.7287	6	1.7394	14	1.7581	6	1.7188
$B; DCA$	5	1.7453	3	1.7955	7	1.7339	3	1.8211	7	1.6487
$B; CDA$	7	1.7393	4	1.7941	8	1.7244	7	1.8030	8	1.6474
$B; DAC$	9	1.6953	7	1.7356	11	1.6923	6	1.8034	13	1.5888
$B; CAD$	12	1.6775	8	1.7316	14	1.6631	15	1.7431	14	1.5849
$B; ADC$	13	1.6706	11	1.7102	13	1.6682	11	1.7800	19	1.5634
$B; ACD$	15	1.6579	12	1.7073	18	1.6472	16	1.7368	20	1.5606
$C; DBA$	10	1.6945	13	1.6652	9	1.7209	5	1.8035	9	1.6391
$C; BDA$	11	1.6810	17	1.6447	10	1.7094	10	1.7830	10	1.6362
$C; DAB$	14	1.6634	14	1.6557	12	1.6842	9	1.7940	15	1.5821
$C; BAD$	19	1.6203	21	1.5852	16	1.6484	17	1.7235	16	1.5739
$C; ADB$	18	1.6397	19	1.6325	15	1.6603	12	1.7708	21	1.5569
$C; ABD$	21	1.6088	22	1.5819	20	1.6346	18	1.7202	22	1.5511
$D; CBA$	16	1.6487	15	1.6547	17	1.6478	19	1.6645	11	1.6285
$D; BCA$	17	1.6411	18	1.6355	19	1.6458	20	1.6620	12	1.6270
$D; CAB$	20	1.6179	16	1.6452	21	1.6118	21	1.6553	17	1.5717
$D; BAC$	23	1.5925	23	1.5786	22	1.6052	23	1.6449	18	1.5674
$D; ACB$	22	1.6000	20	1.6233	23	1.5973	22	1.6500	23	1.5479
$D; ABC$	24	1.5816	24	1.5753	24	1.5925	24	1.6422	24	1.5448

Table D.1.1 Ranking of 4-GSeP scenarios of $(1;3)$-type for $n = 4$, $k = 3$, and $T = 10^9$ runs.

scenario	$v^{(1)} = \left(\frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1\right)$		$v^{(3)} = \left(\frac{1}{4}, \frac{9}{16}, \frac{5}{8}, 1\right)$		$v^{(5)} = \left(\frac{1}{4}, \frac{7}{8}, \frac{15}{16}, 1\right)$	
	rank	total sum	rank	total sum	rank	total sum
$A;DCB$	1	65.9782	1	66.0832	1	64.7201
$A;CDB$	2	65.9569	3	66.0477	2	64.7151
$A;DBC$	3	65.9437	2	66.0742	3	64.7144
$A;CBD$	4	65.9011	5	66.0024	4	64.7044
$A;BDC$	5	65.8897	4	66.0302	5	64.7041
$A;BCD$	6	65.8679	6	65.9937	6	64.6989
$DA;CB$	7	64.2938	7	64.4590	7	62.2789
$CA;DB$	8	64.2589	9	64.3922	9	62.2614
$DA;BC$	9	64.2451	8	64.4463	8	62.2709
$CA;BD$	10	64.1773	11	64.3244	10	62.2459
$AD;CB$	11	64.1386	12	64.3039	13	62.1237
$AC;DB$	12	64.1339	15	64.2875	15	62.1128
$BA;DC$	13	64.1055	10	64.3529	11	62.2330
$AD;BC$	14	64.0899	14	64.2911	14	62.1157
$BA;CD$	15	64.0719	13	64.2976	12	62.2257
$AC;BD$	16	64.0523	17	64.2197	16	62.0973
$AB;DC$	17	64.0266	16	64.2604	17	62.0915
$AB;CD$	18	63.9931	18	64.2051	18	62.0842
$B;DCA$	19	63.7975	19	63.4331	19	60.1723
$B;CDA$	20	63.7775	20	63.4002	20	60.1680
$B;DAC$	21	63.7076	21	63.3575	21	60.0659
$B;CAD$	22	63.6696	22	63.2941	22	60.0577
$B;ADC$	23	63.6040	23	63.2557	23	59.9600
$B;ACD$	24	63.5836	24	63.2221	24	59.9556
$C;DBA$	25	62.0487	25	62.9844	25	59.8336
$C;BDA$	26	62.0001	26	62.9441	26	59.8243
$C;DAB$	27	61.9915	27	62.9175	27	59.7321
$C;BAD$	28	61.8964	28	62.8391	28	59.7147
$C;ADB$	29	61.8918	29	62.8167	29	59.6269
$C;ABD$	30	61.8420	30	62.7754	30	59.6175
$D;CBA$	31	60.6186	31	60.7149	31	59.5053
$D;BCA$	32	60.5896	32	60.7073	32	59.5005
$D;CAB$	33	60.5629	33	60.6508	33	59.4043
$D;BAC$	34	60.5055	34	60.6358	34	59.3954
$D;ACB$	35	60.4826	35	60.5828	35	59.3036
$D;ABC$	36	60.4526	36	60.5750	36	59.2987

Table D.1.2 Ranking of all 36 scenarios of 4-GSeP for $n = 96$, $k = 48$, $v^{(1)} = \left(\frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1\right)$, $v^{(3)} = \left(\frac{1}{4}, \frac{9}{16}, \frac{5}{8}, 1\right)$, $v^{(5)} = \left(\frac{1}{4}, \frac{7}{8}, \frac{15}{16}, 1\right)$, and $T = 10^8$ runs.

scenario	$v^{(2)} = \left(\frac{1}{4}, \frac{5}{16}, \frac{15}{16}, 1\right)$		$v^{(4)} = \left(\frac{1}{4}, \frac{5}{16}, \frac{3}{8}, 1\right)$	
	rank	total sum	rank	total sum
$A; DCB$	1	66.1904	1	67.4249
$A; CDB$	2	66.1858	3	67.3553
$A; DBC$	3	66.1002	2	67.4097
$A; CBD$	4	66.0916	5	67.2694
$A; BDC$	5	66.0095	4	67.3267
$A; BCD$	6	66.0049	6	67.2549
$B; DCA$	7	65.5883	7	66.8123
$B; CDA$	8	65.5840	9	66.7441
$B; DAC$	9	65.4819	8	66.7792
$B; CAD$	10	65.4737	11	66.6433
$B; ADC$	11	65.3760	10	66.6821
$B; ACD$	12	65.3716	12	66.6118
$DA; CB$	13	64.6568	13	66.6021
$CA; DB$	14	64.6544	15	66.5740
$DA; BC$	15	64.5303	14	66.5812
$CA; BD$	16	64.5211	20	66.4407
$AC; DB$	17	64.5058	16	66.5286
$AD; CB$	18	64.5016	19	66.4469
$AD; BC$	19	64.3751	21	66.4260
$AC; BD$	20	64.3725	23	66.3952
$BA; DC$	21	64.2583	17	66.5191
$BA; CD$	22	64.2510	22	66.4051
$AB; DC$	23	64.2337	18	66.4945
$AB; CD$	24	64.2264	24	66.3805
$C; DBA$	25	61.1199	25	66.2245
$C; DAB$	26	61.1020	26	66.2065
$C; BDA$	27	61.0400	27	66.1446
$C; ADB$	28	61.0066	28	66.1112
$C; BAD$	29	60.9410	29	66.0456
$C; ABD$	30	60.9240	30	66.0285
$D; CBA$	31	60.7879	31	61.8954
$D; CAB$	32	60.7700	33	61.8789
$D; BCA$	33	60.7120	32	61.8829
$D; ACB$	34	60.6787	35	61.8510
$D; BAC$	35	60.6170	34	61.8532
$D; ABC$	36	60.6001	36	61.8378

Table D.1.3 Ranking of all 36 scenarios of 4-GSeP for $n = 96$, $k = 48$, $v^{(2)} = \left(\frac{1}{4}, \frac{5}{16}, \frac{15}{16}, 1\right)$, $v^{(4)} = \left(\frac{1}{4}, \frac{5}{16}, \frac{3}{8}, 1\right)$, and $T = 10^8$ runs.

Bibliography

[Alt1985] I. Althöfer: *Das Dreihirn – Entscheidungsteilung im Schach*, Falken-Verlag, ComputerSchach & Spiele, Heft 6, pp. 20-22, December 1985. URL: `http://www.3-hirn-verlag.de`. (Cited on page 25.)

[Alt1998] I. Althöfer: *13 Jahre 3-Hirn – Meine Schach-Experimente mit Mensch-Maschinen-Kombinationen*, 3-Hirn-Verlag, 1998. URL: `http://www.3-hirn-verlag.de`. (Cited on page 25.)

[Bae2011] M. Bärthel: *Viele Köche verderben nicht immer den Brei - Ein diskretes Expertenmodell zum Assortment-Problem*, unpublished term paper, Faculty of Mathematics and Computer Science, Friedrich-Schiller-University Jena, personally communicated in February 2011. (Cited on pages 8, 14, 31, 32, and 61.)

[Bor1781] J.-C. de Borda: *Mémoire sur les Élections au Scrutin*. Histoire del l'Académie Royale des Sciences, Paris, 1781. (Cited on page 31.)

[Gra1953] A. De Grazia: *Mathematical Derivation Of An Election System*, Isis, Volume 44, June 1953. (Cited on page 31.)

[Hil2010] A. Hilbert: *Auswahlverfahren mit korrelierten Experten*, diploma thesis, Faculty of Mathematics and Computer Science, Friedrich-Schiller-University Jena, September 2010. (Cited on pages 8, 14, 30, 31, and 84.)

[Kae2010] N. Kästner: *Summentyp-Optimierungs-Probleme mit korrelierten Experten*, diploma thesis, Faculty of Mathematics and Computer Science, Friedrich-Schiller-University Jena, April 2010. (Cited on pages 8, 14, 27, 28, 29, 30, and 84.)

[Kol2004a] S. Kolassa: *Additional Data for Two-Step Shortlisting by Imperfect Experts*, technical report, Faculty of Mathematics and Computer Science, Friedrich-Schiller-University Jena, April 2004. URL: `http://www.minet.uni-jena.de/preprints/kolassa_04/shortlisting_additional_data.pdf`. (Cited on pages 8, 14, 27, and 84.)

[Kol2004b] S. Kolassa: *Multi-Step Shortlisting by Imperfect Experts*, 3-Hirn-Verlag, also doctoral dissertation, Faculty of Mathematics and Computer Science, Friedrich-Schiller-University Jena, April 2004. URL: `http://www.3-hirn-verlag.de`. (Cited on pages 8, 14, 25, 26, 27, 30, and 84.)

[Kru1956] J. B. Kruskal: *On The Shortest Spanning Subtree Of A Graph And The Traveling Salesman Problem*, Proceedings of the American Mathematical Society, Volume 7, Number 1, pp. 48-50, February 1956. (Cited on page 29.)

[Mar2001] B. H. Margolius: *Permutations with Inversions*, Journal of Integer Sequences, Volume 4, Issue 2, Article 01.2.4, 2001. (Cited on page 32.)

[Pri1957] R. C. Prim: *Shortest Connection Networks And Some Generalizations*, Bell System Technical Journal, Volume 36, pp. 1389-1401, November 1957. (Cited on pages 28 and 29.)

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